PROBLEMS ON BILINEAR FORMS

(1) (a) Let $\mathbb{K}$ be a field with $\mathrm{char}(\mathbb{K}) \neq 2$, and let $Q(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$ be homogeneous of degree two. Show that

$$B(\vec{x}, \vec{y}) := \frac{1}{2}\big(Q(x_1 + y_1, \ldots, x_n + y_n) - Q(x_1, \ldots, x_n) - Q(y_1, \ldots, y_n)\big)$$

defines a symmetric bilinear form on $\mathbb{K}^n$. (Note: such Q are called *quadratic forms.*)

(b) Converseley, show that if $B(\vec{x}, \vec{y})$ is a symmetric bilinear form on $\mathbb{K}^n$, then

$$Q(x_1, \ldots, x_n) := B(\vec{x}, \vec{x})$$

gives a homogeneous degree two polynomial, and that $B(\vec{x}, \vec{y})$ is recovered by the procedure in part (a). Use this to give an alternative proof that if $B(\vec{x}, \vec{y})$ is a non-degenerate bilinear form on $\mathbb{K}^n$, then there exists a non-zero vector so that $B(\vec{x}, \vec{x}) \neq 0$. (Recall this was the first step in our proof that symmetric bilinear forms can be diagonalized).

(2) Suppose that $g(v, w)$ is a reflexive bilinear form. Show that it is either symmetric or alternating. (Hint: consider $g(v, g(v, w)x - g(v, x)w)$.)

(3) Compute the indices of inertia for the following matrices:

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & 1 & -2 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & -1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & -3 \end{pmatrix}$$

(4) Let V be a finite-dimensional $\mathbb{K}$-vector space, and let $g(v, w)$ be a non-degenerate, symmetric bilinear form. Let $W \subset V$ be a subspace, then show that $(W^{\perp})^{\perp} = W$.

(5) Show that $g(A, B) := \mathrm{Tr}(A^T B)$ is a symmetric, positive definite bilinear form on $M_n(\mathbb{R})$.

(6) Let $g(v, w)$ be a (fixed) non-degenerate bilinear form on a finite-dimensional $\mathbb{K}$-vector space V.

(a) Show that the assignment $A \mapsto g(Av, w)$ defines a vector space isomorphism $\mathrm{End}(V) \cong \mathrm{Bilin}(V \times V, \mathbb{K})$.

(b) Show that for each $A \in \mathrm{End}(V)$, there exists a unique linear transformation ${}^tA : V \to V$ so that $g(Av, w) = g(v, {}^tAw)$. This is the *adjoint* or *transpose* with respect to g.

(c) Show that if $g(v, w)$ is the dot product on $\mathbb{K}^n$, then ${}^tA = A^T$.

(d) Let $g(v, w)$ be given on $\mathbb{R}^2 \times \mathbb{R}^2$ by the 2×2 matrix $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. Find ${}^tA$ for the linear transformation given by $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(7) Show that any hermitian form $h(v, w)$ on a finite-dimensional $\mathbb{C}$-vector space can be written as $h(v, w) = g(v, w) + if(v, w)$, where g, f are bilinear forms on the underlying $\mathbb{R}$-vector space, g is symmetric, and f is alternating.

(8) Let $g(v, w)$ be an inner product on a finite-dimensional $\mathbb{R}$-vector space V (*i.e.* a symmetric, positive-definite bilinear form). We say that a linear transformation $U : V \to V$ is *unitary* if ${}^tU = U^{-1}$. Show that $V = V_1 \perp \cdots \perp V_k$ where each V_i is U-invariant and has dimension one or two.

Further, show that if we take a g-orthonormal basis on a 2-dimensional V_i, then in this basis U is given by a matrix of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$