MATH 534 – HOMEWORK 3

(1) Let G be a group, and let $H < G$ and $K < G$ be two proper subgroups. Show that $G \neq H \cup K$ (as sets).

(Hint: this is an odd-numbered problem in the book, and as such has its "answer" in the back of the book. That answer is missing **a lot** of details, *i.e.* they don't justify any of their assertions. I'd probably give it a C+/B- at best. Fill in the missing logical steps, or better yet, provide your own proof without looking at theirs!)

(2) Let G be a group which contains two distinct elements $a, b \in G$ which commute (*i.e.* $ab = ba$) and satisfy $|a| = 2 = |b|$. Prove that G contains a subgroup $H \leq G$ of order 4.

(3) In this problem, we'll introduce and analyze an infinite group which has elements of finite order. A rectangular array of numbers of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called a 2×2 matrix. Two such matrices are equal if and only if all of their the corresponding entries are equal. We'll denote the set of all 2×2 matrices with entries in $\mathbb{R}$ by $M_2(\mathbb{R})$. There is a binary operation, called matrix multipliation, on $M_2(\mathbb{R})$ defined by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}$$

(a) Let $GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \mid ad - bc \neq 0 \right\}$. Prove that $(GL_2(\mathbb{R}), \cdot)$ is a group. (hint: try $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} = \frac{1}{ad-bc} \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, where in the latter equation we set $\alpha \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{smallmatrix}\right)$ for a *scalar* $\alpha \in \mathbb{R}$)

(b) Is $GL_2(\mathbb{R})$ abelian? If so, prove it, and if not, give an example indicating why it isn't.

(c) Prove that $SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \mid ad - bc = 1 \right\}$ is a subgroup of $GL_2(\mathbb{R})$.

(d) Compute the orders of $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

(4) Show that a group of order 3 must be cyclic.

(5) Let G be a group and let $g \in G$ have infinite order, *i.e.* $|g| = \infty$. For $k, l \in \mathbb{Z}$, show that the cyclic subgroups generated by g^k and g^l are equal if and only if $k = \pm l$, (*i.e.* prove that $\langle g^k \rangle = \langle g^l \rangle \Leftrightarrow k = \pm l$).