ON THE QUANTUM TYPE C SPIDER

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A dissertation submitted to the faculty at the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the College of Arts and Sciences.

Chapel Hill
2020

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ABSTRACT

Logan Tatham: On the Quantum Type C Spider
(Under the direction of David E.V. Rose)

In this thesis, we make significant progress towards finding a diagrammatic description of the category $\mathbf{Rep}(U_q(\mathfrak{sp}_{2n}))$ of representations of the quantum group $U_q(\mathfrak{sp}_{2n})$. A diagrammatic description of $U_q(\mathfrak{g})$ for other Lie types $\mathfrak{g}$ is known:

- $\mathfrak{g} = \mathfrak{sl}_2$ (Rumer-Teller-Weyl 1932)
- $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_4, \mathfrak{g}_2$ (Kuperberg, 1996)
- $\mathfrak{g} = \mathfrak{sl}_n$ (Cautis-Kamnitzer-Morrison, 2012)

but beyond this, nothing is known. Diagrammatic descriptions of $U_q(\mathfrak{g})$ are useful for many reasons, including a diagrammatic description of the Reshetikhin-Turaev link invariants, which then lend towards categorification.

In this work, we introduce a diagrammatic category $\mathbf{Web}(\mathfrak{sp}_6)$ which we conjecture is equivalent to the category of tensor products of fundamental $U_q(\mathfrak{sp}_6)$ representations, $\mathbf{FundRep}(U_q(\mathfrak{sp}_6))$. We define a functor $\mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{FundRep}(U_q(\mathfrak{sp}_6))$ which is full and essentially surjective, and conjecture that it is faithful. Further, we prove that $\mathbf{Web}(\mathfrak{sp}_6)$ satisfies some necessary properties of being equivalent to $\mathbf{FundRep}(U_q(\mathfrak{sp}_6))$, including having the correct categorical trace and that $\mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_6)}(\emptyset) = \mathbb{C}(q)$ (where $\emptyset$ is the monoidal unit of $\mathbf{Web}(\mathfrak{sp}_6)$). We also define a braiding on $\mathbf{Web}(\mathfrak{sp}_6)$; together with the above theorems, this gives an explicit construction of the corresponding link invariants. Finally, we have made some progress towards finding a presentation of $\mathbf{Web}(\mathfrak{sp}_{2n})$ for general n.

To May Priegel, for her unwavering love and support. To my family, for their unwavering love. To everyone who has had similar struggles as me in their life and didn't think they could get through them.

ACKNOWLEDGEMENTS

I would like to say **thank you** to the following people:

- My excellent advisor, David Rose, for so much invaluable help in this work, without which, none of this would be possible. I'm glad he took a risk on me by picking me as his first student, especially when there were so many other strong students he could have taken. For doing more than just pushing me when I needed it, but also being sensitive and patient with me when I needed that. And finally, for giving me quite a bit of freedom in choosing what to research; I hope he will forgive me for the lack of categorification in this work!
- My partner, May Priegel, for her unwavering love and support, not only with my research, but of everything I do, and for always believing in me.
- My parents, Kim and Glen Larson, for giving me a great home while growing up, and for always being willing to listen to me while I went through challenges.
- My grandparents, Dale and Mary Ellen Peterson, for giving me a great childhood and teaching me how to work and feel confident.
- My birth dad, Rich Tatham, for helping me have a great childhood and for being inspirational to me in turning his life around.
- My siblings, Lindsay, Lexy, Liberty, for being some of my best friends throughout my life, and for knowing me so well.
- My friends Russell Ollerton, Spenser Squire, for introducing me to yerba mate and being my best friends.
- Alex Safsten, Erica Kiefer, Zac Lowell.
- Russ Arnold, Austin Ferguson, Sam Jeralds, Gonzalo Cazes, Dmitro Golovonich, Marc Besson, and probably others, for making the math department a great place.

- Bill Reel and Radio Free Mormon, for their excellent podcasts which gave me support and clarity in areas no one else could. They both feel like close friends.
- The math and economics departments at Brigham Young University, especially Denise Halverson and James McDonald for their research mentorship.
- The department of math at the University of North Carolina at Chapel Hill, for fostering a great environment in which to do graduate work.
- All the other schools I've attended, but especially Jollyville Elementary and their Special Education program.
- UNC CAPS.
- The countless undergrads at UNC who gave me extra meal swipes, which amounted to me having free lunch and dinner almost daily for about 3 years
- Duke Cannon, for sending me free stuff because I emailed them and asked them to.

I would like to say **no thank you** to the following:

- The current worldwide coronavirus pandemic, for infecting too many of my family members.
- That time I broke my tooth on a smoothie and it cost me a ton to fix it.
- That time a bus ran into my legally parked and stationary car.
- That time a health clinic erroneously charged me a small fortune for a treatement I had refused.

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CHAPTER 1

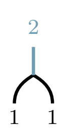
Introduction

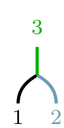
The goal of this thesis is to find a diagrammatic description of the category of representations of the quantum group $U_q(\mathfrak{sp}_6)$. This can be viewed as a categorical analogue of finding a presentation of a group via generators and relations. This problem has been solved in other types. There exist diagrammatic presentations of $\mathbf{FundRep}(U_q(\mathfrak{g}))$ in the cases:

- $\mathfrak{g} = \mathfrak{sl}_2$, [29], 1932
- $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_4, \mathfrak{g}_2$ [17], 1997
- $\mathfrak{g} = \mathfrak{sl}_n$ [4], 2012.

It has remained a difficult open problem to extend these results to other Lie types. In this work, we make the first substantial progress towards solving this problem in other types. We begin by summarizing our results before detailing the relevant background in Chapter 2. We define a category $\mathbf{Web}(\mathfrak{sp}_6)$ which we conjecture is equivalent to the category of (direct sums of tensor products of) fundamental $U_q(\mathfrak{sp}_6)$ representations, $\mathbf{FundRep}(U_q(\mathfrak{sp}_6))$; we further prove a number of results in support of this conjecture.

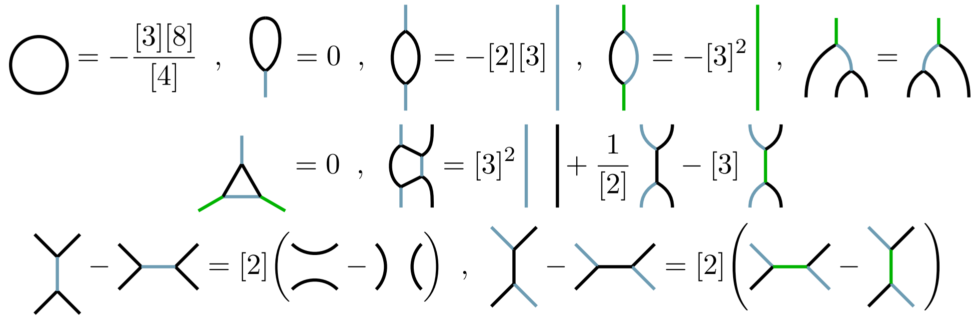
Our category $\mathbf{Web}(\mathfrak{sp}_6)$ is presented using diagrammatic language for monoidal categories, which is reviewed in Section 2.4; it is the (strictly pivotal) $\mathbb{C}(q)$-linear category generated by the self-dual objects $\{1, 2, 3\}$ and with morphisms generated by



$\in \mathrm{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(1 \otimes 1, 2)$,


$\in \mathrm{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(1 \otimes 2, 3)$

modulo the relations:



We prove that there is a functor

$$\Psi : \mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{FundRep}(U_q(\mathfrak{sp}_6))$$

which sends object i to the fundamental representation with highest weight ω_i, and morphisms to certain module maps defined in Section 2.3. While we have not yet proved that Ψ is an equivalence of categories, we have proved that $\mathbf{Web}(\mathfrak{sp}_6)$ satisfies many necessary properties of being equivalent to $\mathbf{FundRep}(U_q(\mathfrak{sp}_6))$. These are:

1. The functor Ψ is full and essentially surjective

2. There is a functor from $\mathbf{Web}(\mathfrak{sp}_6)$ to (matrices of) $\mathbf{Web}(\mathfrak{sp}_4)$ giving a combinatorial analogue of the restriction functor $\mathbf{Rep}(U_q(\mathfrak{sp}_6)) \to \mathbf{Rep}(U_q(\mathfrak{sp}_4))$ arising from the inclusion $U_q(\mathfrak{sp}_4) \hookrightarrow U_q(\mathfrak{sp}_6)$.

3. There are representations of the BMW algebra (which is "Schur-Weyl dual" [10] to the endomorphism algebras of the vector representation) to endomorphism algebras in $\mathbf{Web}(\mathfrak{sp}_{2n})$.

4. The category $\mathbf{Web}(\mathfrak{sp}_6)$ is *ribbon*, and Ψ is a braided monoidal functor.

5. The functor Ψ induces an isomorphism $\mathrm{Tr}(\mathbf{Web}(\mathfrak{sp}_6)) \cong \mathrm{Tr}(\mathbf{Rep}(U_q(\mathfrak{sp}_6)))$.

6. $\dim \mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_6)}(\emptyset) = 1$, so Ψ is faithful when restricted to the monoidal unit $\emptyset$.

Putting facts (3), (5), and (6) together, we obtain an explicit local construction of the quantum $\mathfrak{sp}_6$ link invariant, akin to the Kauffman bracket formulation of the Jones polynomial.

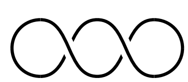
Allow us to pause for a discussion $\mathbf{FundRep}(U_q(\mathfrak{sp}_6))$. This category has objects tensor-generated by (that is, finite tensor products of) fundamental $U_q(\mathfrak{sp}_6)$ representations. One may hope for a presentation of the category $\mathbf{Rep}(U_q(\mathfrak{sp}_6))$ of *all* irreducible, finite dimensional $U_q(\mathfrak{sp}_6)$ representations. However, we would prefer a finite presentation, and a description of $\mathbf{Rep}(U_q(\mathfrak{sp}_6))$ has infinitely many generating objects! While $\mathbf{FundRep}(U_q(\mathfrak{sp}_6))$ does not contain every irreducible finite dimensional $U_q(\mathfrak{sp}_6)$ representation, any such representation is the highest weight irreducible summand of a tensor product of finitely many fundamental representations, so in a sense, $\mathbf{FundRep}(U_q(\mathfrak{sp}_6))$ has enough to "see" everything in $\mathbf{Rep}(U_q(\mathfrak{sp}_6))$. In fact, there is a process to make this formal, called the Karoubi (or idempotent) completion, and we indeed have

$$\mathbf{Kar}(\mathbf{FundRep}(U_q(\mathfrak{g}))) = \mathbf{Rep}(U_q(\mathfrak{g}))$$

where **Kar** denotes the Karoubi completion.

Of course, one may ask why diagrammatic descriptions of $\mathbf{FundRep}(U_q(\mathfrak{g}))$ is useful. Due to Reshetinkin-Turaev [26], $\mathbf{FundRep}(U_q(\mathfrak{g}))$ can be used to construct link invariants in S^3, so $\mathbf{FundRep}(U_q(\mathfrak{g}))$ is braided. For example, when $\mathfrak{g} = \mathfrak{sl}_2$, this gives a diagrammatic description of the Jones polynomial [13]. While extending this to other types may be enough motivation, there is an added benefit of a diagrammatic description.

One old problem of knot theory is the Tait conjecture, posed by Tait in the 1880's [18]. While tabulating knots, he noticed that alternating knot diagrams (meaning diagrams in when, when a strand is followed, it alternates going over and under a crossings) seemed to be minimal; that is, have a minimal number of crossings. This led him to ask if an alternating diagram of a knot is always minimal. As stated, the answer is no; for example, the knot diagram



is alternating, but not minimal since it represents the unknot. However, loosely speaking, if we don't have cases where we can "obviously" untwist the diagram to untwist a crossing, then the answer is yes. The Jones polynomial was introduced in 1984, but once the diagrammatic description of the Jones polynomial was introduced by Kauffman in 1987, the Tait conjecture was proved independently by Kauffman, Murasugi, and Thislewaite [12] [33] [22].

As another example, Khovanov [14] used the diagrammatic description of the Jones polynomial to *categorify* the Jones polynomial. Specifically, given a link $\mathcal{L}$ with diagram $\mathcal{L}_D$, we may take the homology $\mathrm{Kh}(\mathcal{L})$ of a certain (bounded) chain complex of graded vector spaces which then satisfies

$$V_q(\mathcal{L}) = \chi\left(\mathrm{Kh}(\mathcal{L}_D)\right)$$

where V_q denotes the Jones polynomial and χ denotes the Euler characteristic. (In fact, Kh is an invariant of links, so we perhaps ought to write $\mathrm{Kh}(\mathcal{L})$ rather than $\mathcal{L}_{\mathrm{D}}$). The construction of the chain complex arising from $\mathcal{L}_D$ depends heavily on the diagrammatic presentation of $\mathbf{FundRep}(U_q(\mathfrak{sl}_2))$. While the Jones polynomial is a strong invariant, Khovanov homology has been shown to contain even more topological information. For example, Khovanov homology is an unknot detector, meaning $\mathrm{Kh}(\mathcal{L}) = \mathrm{Kh}(O)$ if and only if $\mathcal{L} \sim O$ (where O denotes the unknot; it remains an open question as to whether this is true for the Jones polynomial). Another example of the usefulness of Khovanov homology is the masterful use of it by Rasmussen [24] to give a rather tractable proof of the Milnor conjecture.

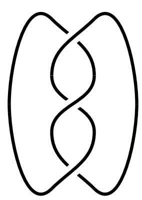
Diagrammatic descriptions of $\mathbf{FundRep}(U_q(\mathfrak{g}))$ are a useful tool in representation theory as well. For example, Elias has used diagrammatic descriptions to prove that $\mathbf{FundRep}(U_q(\mathfrak{sl}_n))$ is a *cellular* category [7]. As another example, Elias has used webs to give a proof of the geometric Satake equivalence; furthermore, Elias uses the web-based proof to introduce a quantum analogue of the geometric Satake equivalence [8].

CHAPTER 2

Background

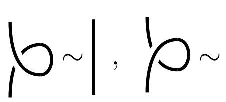
2.1 Knots, Links, and the Jones Polynomial

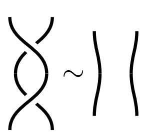
As motivation for the diagrammatic formalism to follow, we begin by reviewing the basics of knot theory. A *knot* is an embedding $S^1 \hookrightarrow S^3$ considered up to the equivalence that $\mathcal{K}_1 \sim \mathcal{K}_2$ if there exists a smooth isotopy $h_t : I \times S^3 \to S^3$ such that$h_0 = id_{S^3}$ and $h_1(\mathcal{K}_1) = \mathcal{K}_2$. A *link* is a disjoint union of knots. More often than not, we study knots and links via their *diagrams*. A link diagram is a projection of a link to the plane (put in general position, so we neither have a tangency of strands nor a crossing of more than two strands) where at each vertex, we remember whether a strand is over or under another, and decorate them accordingly. As an example,

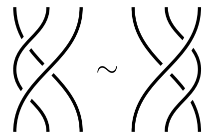


is a diagram of a trefoil knot. Several link diagrams may represent the same link, and it is not always obvious when different link diagrams represent the same link! Fortunately, we have the following theorem, due to Reidemeister [25].

Theorem 2.1.1. Let $\mathcal{D}_1$, $\mathcal{D}_2$ be link diagrams, representing links $\mathcal{L}_1$, $\mathcal{L}_2$, respectively. Then $\mathcal{L}_1 \sim \mathcal{L}_2$ if and only if $\mathcal{D}_1, \mathcal{D}_2$ differ from (a finite sequence) of the following three local moves:

R1: 

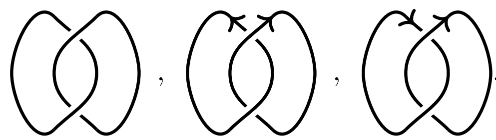
R2: 

R3: 

We call the moves R1, R2, and R3 the first, second, and third Reidemeister moves, respectively.

Thus, studying links up to ambient isotopy is equivalent to studying link diagrams up to the three Reidemeister moves.

As will become apparent, the theory lends itself better to the study of *oriented links*, which are links where we have assigned each component an orientation. For example, below we have an unoriented *Hopf link* followed by two different orientations of a Hopf link:



It is believable enought that the two differently oriented Hopf links shown above are not equivalent; indeed, we soon will have tools to assure us of this. One may conjecture that in a case of a knot, the orientation doesn't matter (*i.e.* if we switch the orientation of a knot, we get an equivalent knot). However, this is not true; for a counterexample, see the knot 8_{17} in the Rolfsen knot table [27].

Given an oriented link diagram, it will be useful to distinguish different types of crossings. We define the sign of a crossing as follows:

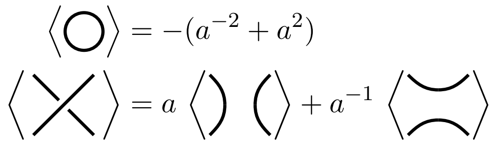
 is a *positive* crossing

 is a *negative* crossing.

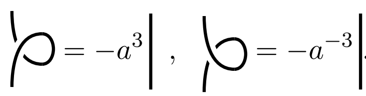
We are ready to define an important invariant in knot theory: the Jones polynomial. Here, we will define and discuss the Jones polynomial. There are several paths to arrive to the Jones polynomial; we shall follow the path of Kauffman, as we shall see it will relate closely to our work.

Definition 2.1.2. Define the *Kauffman bracket* of a link diagram (written $\langle \mathcal{L}_{\mathcal{D}} \rangle$) by forgetting the orientation and applying the formal rules on a diagram:

$$\langle \mathcal{L}_1 \amalg \mathcal{L}_2 \rangle = \langle \mathcal{L}_1 \rangle \cdot \langle \mathcal{L}_2 \rangle$$



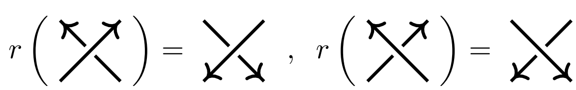
One can verify that the Kauffman bracket (a polynomial in a) is invariant under the second and third Reidemeister moves, and



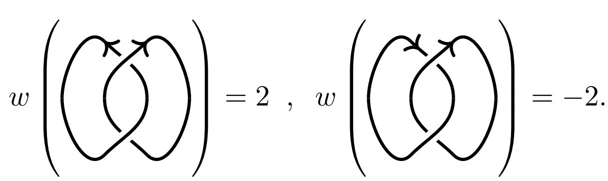
So unfortunately, the Kauffman bracket isn't invariant under all three Reidemeister moves. However, we can correct this problem as follows. Define the *write* of an oriented link diagram $\mathcal{D}$ as

$$w(\mathcal{D}) = n_+(\mathcal{D}) - n_-(\mathcal{D})$$

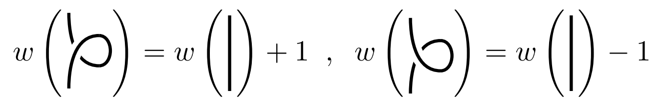
where $n_+(\mathcal{D}), n_-(\mathcal{D})$ are the number of positive and negative crossings of $\mathcal{D}$, respectively. Given a knot diagram, the writhe is well-defined. Suppose we have oriented the knot. Then, if $r(\mathcal{D})$ is the knot diagram $\mathcal{D}$ with orientation reversed, we have



so reversing orientation sends positive crossings to positive crossings and negative crossings to negative crossings. However, writhe is not well-defined on link diagrams; for example, the writhes of the two differently-oriented Hopf links are



Note that the writhe is invariant under (any orientation of) the second and third Reidemeister moves. However, the writhe isn't invariant under the first Reidemeister move:



for any orientation of the strand. However, we can combine the Kauffman bracket and writhe in such a way to counteract their failure to be invariant under the Reidemeister 1 move.

Theorem 2.1.3. The polynomial

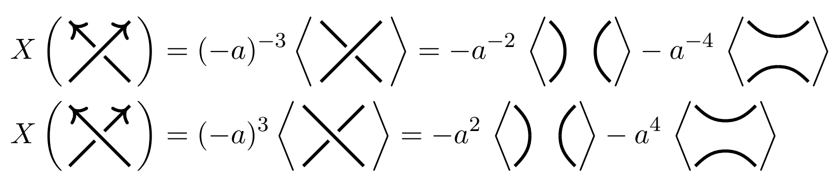
$$X(\mathcal{L}) = (-a)^{-3w(\mathcal{L}_{\mathcal{D}})}\langle \mathcal{L}_{\mathcal{D}} \rangle$$

is invariant under all three Reidemeister moves, and thus an inviariant of a link. In fact, for any link, $X(\mathcal{L}) \in \mathbb{Z}[a^{-2}, a^2]$.

Proof. Because the writhe and bracket polynomials are invariant under the second and third Reidemeister moves, so is X. Suppose $\mathcal{L}_{\mathcal{D}}'$ differs from $\mathcal{L}_{\mathcal{D}}$ by one Reidemeister 1 move (say by a positive twist). Then $w(\mathcal{L}_{\mathcal{D}}') = w(\mathcal{L}_{\mathcal{D}}) + 1$ and $\langle \mathcal{L}_{\mathcal{D}} \rangle = -a^3 \langle \mathcal{L}_{\mathcal{D}} \rangle$. Hence,

$$X(\mathcal{L}_{\mathcal{D}}') = (-a)^{-3(w(\mathcal{L}_{\mathcal{D}})+1)}\left(-a^3\langle \mathcal{L}_{\mathcal{D}} \rangle\right) = (-a)^{-3}(-a)^{-3w(\mathcal{L}_{\mathcal{D}})}(-a^3)\langle \mathcal{L}_{\mathcal{D}} \rangle = X(\mathcal{L}_{\mathcal{D}}).$$

To see that $X(\mathcal{L}) \in \mathbb{Z}[a^{-2}, a^2]$, note that locally, we have



and the circle evaluates to $-a^{-2} - a^2$. Since at each crossing and each circle, we have all even powers of a, $X(\mathcal{L})$ has all even powers of a. $\square$

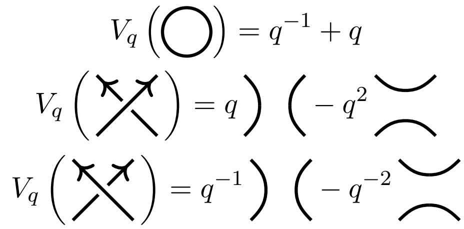
We then may define

Definition 2.1.4. Let $\mathcal{L}$ be an oriented link. Define its *Jones polynomial* $V_q(\mathcal{L})$ by

$$V_q(\mathcal{L}) = X(\mathcal{L})\mid_{q=-a^{-2}}.$$

Alternatively, we may more directly define the Jones polynomial by the rules

$$V_q(\mathcal{L}_1 \amalg \mathcal{L}_2) = V_q(\mathcal{L}_1) \cdot V_q(\mathcal{L}_2)$$



(2.1.1)

Experts may wonder why we've bothered to include the Kauffman bracket rather than jumping right to the above definition of the Jones polynomial (or bothered to distinguish them at all). As we shall see, the Jones polynomial is a special case of our work, and the conventions used in the Kauffman bracket make the fit a bit clearer. For example, the circle value of $-a^{-2} - a^2$ may seem unnatural, but we'll see it follows from a choice of pivotal structure on a monoidal category.

2.2 Quantum Groups

One reason we care about the representation theory of quantum groups is that they lead to link invariants. For example, the definition in Equation (2.1.1) may be interpreted as describing relations of certain morphisms between representation of the *quantum group* $U_q(\mathfrak{sl}_2)$. Due to Reshetikhin-Turaev[1], the category of representations of a quantum group admits a certain diagrammatic presentation in which each "piece" of a link diagram may be interpreted as certain morphisms between representations. A closed link may then be interpreted as a morphism from $\mathbb{C}(q)$ to itself, and some theory in this section tells us that morphism must be an element of $\mathbb{C}(q)$. Thus, every closed link diagram gives us an element of $\mathbb{C}(q)$; when $\mathfrak{g} = \mathfrak{sl}_2$, this element is exactly the Jones polynomial.

To understand quantum groups, it helps to first understand Lie algebras.

Definition 2.2.1. A *Lie algebra* $\mathfrak{g}$ is a vector space $\mathfrak{g}$ over a field $\Bbbk$ (if $\Bbbk = \mathbb{C}$, we say $\mathfrak{g}$ is a complex Lie algebra) along with an operator $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the bracket) which satisfies:

- $[-,-]$ is bilinear

[1]We shall come back to this in Theorem 2.4.9 once we have finished building the vocabulary to make everything precise

- $[x, x] = 0$ for all $x \in \mathfrak{g}$
- The Jacobi identity: for all $x, y, z \in \mathfrak{g}$,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Given Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, a *Lie algebra homomorphism* $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map which satisfies $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}_1$.

Note that if $\Bbbk$ is not of characteristic 2, these imply $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.

A Lie algebra is *abelian* if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

An *ideal* of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{i} \subset \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{i}] \subset \mathfrak{i}$.

Finally, a Lie algebra is *simple* if it is non-abelian and has no non-trivial ideals. A *semisimple* Lie algebra is the finite direct sum of simple Lie algebras.

A simple complex Lie algebra can be described in terms of its *Cartan matrix*, an $n \times n$ matrix (a_{ij}) with[2] $a_{ii} = 2$, $a_{ij} = 0$ if $|i - j| > 1$, and $a_{ij} \in \{-1, -2, -3\}$ for $|i - j| = 1$. Then $\mathfrak{g}$ is generated by the $3n$ elements E_i, F_i, H_i with $i = 1, \ldots, n$ subject to the *Serre relations*:

- $[H_i, H_j] = 0$
- $[E_i, F_j] = \delta_{ij} H_i$
- $[H_i, E_j] = a_{ij} E_j$
- $[H_i, F_j] = -a_{ij} F_j$
- $\mathrm{ad}_{e_i}^{1-a_{ij}} e_j = 0$
- $\mathrm{ad}_{f_i}^{1-a_{ij}} f_j = 0$

where $\mathrm{ad}_x y = [x, y]$. Finally, the subalgebra of $\mathfrak{g}$ generated by the H_i is denoted $\mathfrak{h}$, and called the *Cartan subalgebra*.

[2]These are necessary, not sufficient properties of a Cartan matrix

Definition 2.2.2. Let V be a vector space. Then $\mathfrak{gl}(V)$ is the space of linear maps of V to itself, made into a Lie algebra with bracket defined by $[f, g] = fg - gf$.

A *Lie algebra representation* is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

The (finite dimensional) representation theory of simple complex Lie algebras is well-understood. First, we recall some definitions and fix conventions; we recommend [11] [9] for the remaining details.

To each simple complex Lie algebra, there is a weight lattice $\Lambda \subset \langle^*$, which helps encode the representations. Specifically, the action of $\langle$ on a representation V breaks into weight spaces V_λ where $\mathfrak{h}$ acts on V_λ via $H_i(v) = \lambda(H_i)v$. The weights of the *adjoint representation* (given by $x \mapsto \mathrm{ad}_x$) are called the *roots* of Λ. We fix a collection $\alpha_1, \ldots, \alpha_n$ of *positive roots*. Then Λ has a partial order, where $\lambda_1 \succeq \lambda_2$ if $\lambda_1 - \lambda_2$ is the positive integral sum of positive roots.

The (finite-dimensional) irreducible representations of $\mathfrak{g}$ are classified by the following theorem:

Theorem 2.2.3. Let V be a finite-dimensional irreducible representation of $\mathfrak{g}$. Then V has a highest weight vector with weight $\lambda \in \Lambda^+$; furthermore, V is determined (up to isomorphism) by λ. Finally, for each dominant weight λ, there exists an irreducible representation of highest weight λ.

One important result which we will use often is Schur's Lemma, which classifies maps between irreducible representations.

Lemma 2.2.4. Let V_1, V_2 be irreducible representations. Then if V_1 and V_2 are not isomorphic, $\mathrm{Hom}_\mathfrak{g}(V_1, V_2) = 0$; if V_1, V_2 are isomorphic, $\mathrm{Hom}_\mathfrak{g}(V_1, V_2) = \{c \cdot id_{V_1} : c \in \mathbb{C}\}$.

There is a basis of dominant integral weights $\omega_1, \ldots, \omega_n$ such that each dominant integral weight λ can be written as a non-negative integral sum of the ω_i; we call the ω_i *fundamental weights*. We write $\Gamma_{a_1,\ldots,a_n}$ for the irreducible representation of highest weight $a_1\omega_1 + \cdots + a_n\omega_n$ (each $a_i \geq 0$). Thus, the finite-dimensional irreducible representations of $\mathfrak{g}$ are exactly the set of $\Gamma_{a_1,\ldots,a_n}$ where each $a_i \geq 0$.

The universal enveloping algebra $U(\mathfrak{g})$ is an associative algebra which has the same representation theory as $\mathfrak{g}$. The universal enveloping algebra is a *Hopf algebra*, which dictates how to act on tensor products, duals, and a trivial representation. Specifically, we have the maps $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $S : U(\mathfrak{g}) \to U(\mathfrak{g})$, and $\epsilon : U(\mathfrak{g}) \to \mathbb{C}$ defined by

- $\Delta(x) = x \otimes 1 + 1 \otimes x$

- $S(x) = -x$

- $\epsilon(x) = 0$

Together, this endows $\mathbf{Rep}(U(\mathfrak{g}))$ with the structure of a monoidal pivotal category (we review this definition in 2.4. Furthermore, for representations $V \otimes W$, we have that the map $\beta_{V \otimes W} : V \otimes W \to W \otimes V$ given by $\tau(v \otimes w) = w \otimes v$ (extended linearly) is an isomorphism. Because $\beta_{W \otimes V} \circ \beta_{V \otimes W} = id_{V \otimes W}$, this gives $\mathbf{Rep}(U(\mathfrak{g}))$ the structure of a symmetric monoidal category. For the purpose of computing useful link invariants, we wish to have a braided monoidal category which is not symmetric! Thus, we introduce the *quantum group*, a deformation of the universal enveloping algebra whose representation category is braided, monoidal, pivotal, but *not* symmetric.

First, we will need to make a quick definition.

Definition 2.2.5. Let $n \in \mathbb{Z}$. Define the *quantum integer* $[n] \in \mathbb{Z}[q^{-1}, q]$ by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

When $n > 0$, we have that $[n] = q^{-n+1} + q^{-n+3} + \cdots + q^{n-3} + q^{n-1}$, and $[-n] = -[n]$. For a positive integer d, define $[n]_d = [n]|_{q=q^d}$. Finally, let

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n][n-1]\ldots[2][1]}{([k][k-1]\ldots[2][1])([n-k][n-k-1]\ldots[2][1])}.$$

Remark 2.2.6. We pause to note a couple useful facts about quantum integers. First, taking the limit as $q \to 1$, we have $\lim_{n \to 1}[n] = n$. Another useful fact is that

$$\frac{[nd]}{[d]} = \frac{\frac{q^{nd}-q^{-nd}}{q-q^{-1}}}{\frac{q^d-q^{-d}}{q-q^{-1}}} = \frac{(q^d)^n - (q^d)^{-n}}{(q^d) - (q^d)^{-1}} = [n]_d.$$

As an example, we can use this to simplify $\frac{[n][2n+2]}{[n+1]}$, because

$$\begin{aligned} \frac{[n][2n+2]}{[n+1]}(q - q^{-1}) &= [2]_{n+1}[n](q - q^{-1}) \\ &= (q^{-n-1} + q^{n+1})(q^n - q^{-n}) \\ &= q^{2n+1} - q^{-2n-1} + q^{-1} - q \end{aligned}$$

$$= ([2n+1]-1)(q-q^{-1})$$

so $\frac{[n][2n+2]}{[n+1]} = [2n+1]-1$; this is an identity we shall come back to.

Now we are prepared to define the quantum group.

Definition 2.2.7. Let $\mathfrak{g}$ be a simple complex Lie algebra of rank n with corresponding Cartan matrix (a_{ij}). The *quantum group* $U_q(\mathfrak{g})$ associated to $\mathfrak{g}$ is the unital $\mathbb{C}(q)$-algebra generated by $X_i^+, X_i^-, K_i, K_i^{-1}$ $(1 \le i \le n)$ modulo relations

- $K_iK_i^{-1} = 1$, $K_i^{-1}K_i = 1$
- $K_iK_j = K_jK_i$
- $K_iX_j^+K_i^{-1} = q_i^{a_{ij}}X_j^+$, $K_iX_j^-K_i^{-1} = q_i^{-a_{ij}}X_j^-$
- $X_i^+X_j^- - X_j^-X_i^+ = \delta_{ij}\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}}$
- For $i \neq j$, $\sum_{m=0}^{1-a_{ij}}(-1)^m \left[{1-a_{ij} \atop m}\right]_{q_i} (X_i^pm)^{1-a_{ij}-m}X_j^{\pm}(X_i^{\pm})^m = 0$.

Here, $q_i := q^{d_i}$ where $\{d_i\}_{i=1}^n$ are the relatively prime positive integers such that (d_ia_{ij}) is symmetric.

The quantum group $U_q(\mathfrak{g})$ is also a Hopf algebra, with the maps

- $\Delta(K_i) = K_i \otimes K_i$, $\Delta(X_i^+) = X_i^+ \otimes K_i + 1 \otimes X_i^+$, $\Delta(X_i^-) = X_i^- \otimes 1 + K_i^{-1} \otimes X_i^-$
- $\epsilon(K_i) = 1$, $\epsilon(X_i^+) = \epsilon(X_i^-) = 0$
- $S(K_i) = K_i^{-1}$, $S(X_i^+) = -X_i^+K_i^{-1}$, $S(X_i^-) = -K_iX_i^-$.

There is a classification theorem of (finite-dimensional) irreducible representations (with a highest weight) of $U_q(\mathfrak{g})$, similar to that of 2.2.3.

Theorem 2.2.8. Every (finite-dimensional) irreducible representation V of $U_q(\mathfrak{g})$ has a highest weight $\lambda \in \Lambda^+$; furthermore, V is determined (up to isomorphism) by λ. Finally, for each $\lambda \in \Lambda^+$, there exists an irreducible represention of highest weight λ.

For a proof, see [5] section 10.

The decomposition of tensor products of irreducible $U_q(\mathfrak{g})$ representations into direct sums of irreducible modules is the same as that of $U(\mathfrak{g})$. [5]

One difference in the representation theory of $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ is that there is a quantum analogue of the dimension of an irreducible representation $\dim_q$, which is an element of $\mathbb{N}[q^{-1}, q]$ rather than $\mathbb{N}$.

Definition 2.2.9. Let V be an irreducible representation, let ρ be the *Weyl vector*, defined as half the sum of positive roots (or, equivalently, the sum of fundamental weights). Write $\rho = \sum_{i=1}^{n} r_i \alpha_i$, and let $K_\rho = K_1^{r_1} \dots K_n^{r_n}$. Then the *quantum dimension* of V is given by

$$\dim_q(V) = \mathrm{tr}(K_\rho).$$

One may also compute the quantum dimension of V with the *quantum Weyl character formula*

$$\dim_q(V) = \prod_{\alpha \in \Phi^+} \frac{[(\lambda + \rho, \alpha)]}{[(\rho, \alpha)]}$$

see [5] section 11 for a proof.

Further, $U_q(\mathfrak{g})$ is a ribbon category (defined in 2.4 below); the standard choice of ribbon element of $U_q(\mathfrak{g})$ is the quantum Casimir element C, which acts on an irreducible representation V of highest weight λ by $q^{-(\lambda, \lambda + 2\rho)}$ where ρ is the Weyl vector, which is the half-sum of positive roots (or equivalently, the sum of fundamental weights).

2.3 Examples of FundRep($U_q(\mathfrak{g})$)

In this section, will explicitly lay out some specifics of the representation theory of $U_q(\mathfrak{g})$ for some examples of $\mathfrak{g}$ which are used in this work.

2.3.1 FundRep($U_q(\mathfrak{sl}_2)$)

The Lie algebra $\mathfrak{sl}_2$ has the 1×1 Cartan matrix (2). Writing $E = X_1^+$ and $F = X_1^-$, we have that $U_q(\mathfrak{sl}_2)$ is the $\mathbb{C}(q)$-algebra generated by E, F, K, K^{-1} modulo relations:

- $KK^{-1} = 1$, $K^{-1}K = 1$
- $KE = q^2 EK$, $KF = q^{-2} FK$
- $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$.

Following the representation theory of $U(\mathfrak{sl}_2)$, let $V = \langle v_+, v_- \rangle$ where v_+, v_- are vectors of weights $+1, -1$, respectively. The generators of $U_q(\mathfrak{sl}_2)$ act on V via

V	v_+	v_-
E	0	v_+
F	v_-	0
K	qv_+	$q^{-1}v_-$

Table 2.1: The fundamental $U_q(\mathfrak{sl}_2)$ representation

The tensor product of V with itself decomposes into irreducible modules as

$$V \otimes V = \operatorname{Sym}_q^2 V \oplus \mathbb{C}(q)$$

where

$$\operatorname{Sym}_q^2 V = \langle v_+ \otimes v_+, q^{-1} v_+ \otimes v_- + v_- \otimes v_+, v_- \otimes v_- \rangle$$

$$\mathbb{C}(q) \cong \textstyle\bigwedge_q^2 V = \langle q v_+ \otimes v_- - v_- \otimes v_+ \rangle$$

where $\operatorname{Sym}_q^2 V$, $\bigwedge_q^2 V$ are quantum analogues of $\operatorname{Sym}^2 V$ and $\bigwedge^2 V$, respecively. By Schur's lemma, there exist module maps $p : V \otimes V \to \mathbb{C}(q)$ and $i : \mathbb{C}(q) \to V \otimes V$, unique up to a scalar. We give these explicitly by

$$p : \begin{cases} v_+ \otimes v_+ & \mapsto 0 \\ v_+ \otimes v_- & \mapsto -1 \\ v_- \otimes v_+ & \mapsto q^{-1} \\ v_- \otimes v_- & \mapsto 0 \end{cases} \qquad \text{and} \qquad i : 1 \mapsto q v_+ \otimes v_- - v_- \otimes v_+.$$

2.3.2 FundRep($U_q(\mathfrak{sp}_4)$)

The algebra $U_q(\mathfrak{sp}_4)$ has Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

By Definition 2.2.7 above, this gives an explicit presentation of $U_q(\mathfrak{sp}_4)$. The $\mathfrak{sp}_4$ weight lattice is spanned by weights ϵ_1, ϵ_2, where $(\epsilon_i, \epsilon_j) = \delta_{ij}$. Type $\mathfrak{sp}_4$ has roots $\pm 2\epsilon_1$, $\pm\epsilon_2$, $\pm\epsilon_1 \pm \epsilon_2$. We have simple roots $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = 2\epsilon_2$. The fundamental weights are given by $\omega_1 = \epsilon_1$, $\omega_2 = \epsilon_1 + \epsilon_2$. The sum of positive roots is given by $\rho = 2\epsilon_1 + \epsilon_2$. Finally, we write $\Gamma_{i,j}$ for the irreducible $U_q(\mathfrak{sp}_4)$ representation of highest weight $i\omega_1 + j\omega_2$.

Denote $V = \Gamma_{1,0}$ and $W = \Gamma_{0,1}$. Let v_1 be the highest weight vector of V, and write $v_2 = F_1 v_1$, $v_3 = F_2 v_2$, and $v_4 = F_1 v_1$. Then the action of $U_q(\mathfrak{sp}_4)$ on V is given explicitly by

V	v_1	v_2	v_3	v_4
E_1	0	v_1	0	v_3
F_1	v_2	0	v_4	0
K_1	qv_1	$q^{-1}v_2$	qv_3	$q^{-1}v_4$
E_2	0	0	v_2	0
F_2	0	v_3	0	0
K_2	v_1	q^2v_2	$q^{-2}v_3$	v_4

Table 2.2: The fundamental $U_q(\mathfrak{sp}_4)$ representation V

As in the classical case, $V \otimes V$ decomposes into irreducible representation

$$V \otimes V = \Gamma_{2,0} \oplus W \oplus \mathbb{C}(q)$$

where $\Gamma_{2,0}$ is a q-deformation of $\mathrm{Sym}^2 V$.

The irreducible representation W has weight vectors $w_{ij} = qv_i \otimes v_j - v_j \otimes v_i$ for $1 \leq i < j \leq 4$ and $w_0 = v_1 \otimes v_4 - v_4 \otimes v_1 + qv_2 \otimes v_3 - q^{-1}v_3 \otimes v_2$. We record the action of $U_q(\mathfrak{sp}_4)$ on W:

W	w_{12}	w_{13}	w_{24}	w_{34}	w_0
E_1	0	0	w_0	0	$[2]w_{13}$
F_1	0	w_0	0	0	$[2]w_{24}$
K_1	w_{12}	q^2w_{13}	$q^{-2}w_{24}$	w_{34}	w_0
E_2	0	w_{12}	0	w_{24}	0
F_2	w_{13}	0	w_{34}	0	0
K_2	q^2w_{12}	$q^{-2}w_{13}$	q^2w_{24}	$q^{-2}w_{34}$	w_0

Table 2.3: The fundamental $U_q(\mathfrak{sp}_4)$ representation W

The copy of the trivial representation in $V \otimes V$ is spanned by the vector

$$z = q^2 v_1 \otimes v_4 - q^{-2} v_4 \otimes v_1 - q v_2 \otimes v_3 + q^{-1} v_3 \otimes v_2.$$

Clearly, z has weight 0, and one may verify that $E_1.z = E_2.z = 0$.

Finally, $\Gamma_{2,0}$ can be seen to be 10 dimensional, having weight spaces $\pm 2L_i$, $\pm L_i \pm L_j$ and a 2-dimensional weight zero space, with weight vectors

- (Weight $\pm 2L_i$) $v_i \otimes v_i$ for $1 \leq i \leq 6$
- (Weight $\pm L_i \pm L_j$) $q^{-1} v_i \otimes v_j + v_j \otimes v_i$ for $1 \leq i < j \leq 4$ with $i + j \neq 5$
- (Weight 0) $q^{-1} v_1 \otimes v_4 + q v_4 \otimes v_1 + v_2 \otimes v_3 + v_3 \otimes v_2$
- (Weight 0) $q^{-1} v_2 \otimes v_3 + q v_3 \otimes v_2$

Note that all the above explicitly gives the decomposition

$$V \otimes V = \Gamma_{2,0} \oplus W \oplus \mathbb{C}(q).$$

By Schur's Lemma, there are module maps $i_{\mathbb{C}(q)} : \mathbb{C}(q) \to V \otimes V$, $i_W : W \to V \otimes V$, $p_{\mathbb{C}(q)} : V \otimes V \to \mathbb{C}(q)$, $p_W : V \otimes V \to W$, unique up to a scaler. We define these maps as follows: $p_{\mathbb{C}(q)}$, p_W are defined as above (so, for instance, $p_W(v_0) = v_1 \otimes v_4 - v_4 \otimes v_1 + q v_2 \otimes v_3 - q^{-1} v_3 \otimes v_2$), and the maps are given by

$$p_{\mathbb{C}(q)} : \begin{cases} v_i \otimes v_j \mapsto 0 \text{ if } i+j \neq 5 \\ v_1 \otimes v_4 \mapsto -q^2 \\ v_2 \otimes v_3 \mapsto q \\ v_3 \otimes v_2 \mapsto -q^{-1} \\ v_4 \otimes v_1 \mapsto q^{-2} \end{cases} \quad \text{and} \quad p_W : \begin{cases} v_i \otimes v_i \mapsto 0 \\ v_i \otimes v_j \mapsto -[2]w_{ij} \text{ if } i < j \\ v_j \otimes v_i \mapsto q^{-1}[2]w_{ij} \text{ if } i < j \\ v_1 \otimes v_4 \mapsto -w_0 \\ v_2 \otimes v_3 \mapsto -qw_0 \\ v_3 \otimes v_2 \mapsto q^{-1}w_0 \\ v_4 \otimes v_1 \mapsto w_0. \end{cases}$$

We may then compute that

$$\begin{aligned}(p_{\mathbb{C}(q)}) \circ i_{\mathbb{C}(q)})(1) &= p_{\mathbb{C}(q)}\left(q^2 v_1 \otimes v_4 - q^{-2} v_4 \otimes v_1 - q v_2 \otimes v_3 + q^{-1} v_3 \otimes v_2\right) \\ &= q^2(-q^2) - q^{-2}(q^2) - q(q) + q^{-1}(-q^{-1}) \\ &= -q^{-4} - q^{-2} - q^2 - q^4 \\ &= -\frac{[2][6]}{[3]}\end{aligned}$$

so that $(p_{\mathbb{C}(q)} \otimes id_V) \circ (id_V \otimes i_{\mathbb{C}(q)}) = id_V$. A similar computation shows that $(id_V \otimes p_{\mathbb{C}(q)}) \circ (i_{\mathbb{C}(q)} \circ id_V) = id_V$.

2.3.3 FundRep($U_q(\mathfrak{sp}_6)$)

The quantum group $U_q(\mathfrak{sp}_6)$ has Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

By Definition 2.2.7 above, this gives an explicit presentation of $U_q(\mathfrak{sp}_6)$.

The $\mathfrak{sp}_6$ weight lattice is spanned by weights $\epsilon_1, \epsilon_2, \epsilon_3$, where $(\epsilon_i, \epsilon_j) = \delta_{ij}$. Type $\mathfrak{sp}_6$ has roots $\pm 2\epsilon_i$, $\pm\epsilon_i \pm \epsilon_j$ for $1 \leq i < j \leq 3$. We have simple roots $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \alpha_3 = 2\epsilon_3$. The fundamental weights are given by $\omega_1 = \epsilon_1$, $\omega_2 = \epsilon_1 + \epsilon_2$, and $\omega_3 = \epsilon_1 + \epsilon_2 + \epsilon_3$. The sum of positive roots is given by $\rho = 3\epsilon_1 + 2\epsilon_2 + \epsilon_3$. Recall that we write $\Gamma_{i,j,k}$ for the irreducible $U_q(\mathfrak{sp}_6)$ representation of highest weight $i\omega_1 + j\omega_2 + k\omega_3$.

Denote the three fundamental representations by $V = \Gamma_{1,0,0}$, $W = \Gamma_{0,1,0}$, and $X = \Gamma_{0,0,1}$. One can check that V is given by:

V	v_1	v_2	v_3	v_4	v_5	v_6
E_1	0	v_1	0	0	0	v_5
F_1	v_2	0	0	0	v_6	0
K_1	qv_1	$q^{-1}v_2$	v_3	v_4	qv_5	$q^{-1}v_6$
E_2	0	0	v_2	0	v_4	0
F_2	0	v_3	0	v_5	0	0
K_2	v_1	qv_2	$q^{-1}v_3$	qv_4	$q^{-1}v_5$	v_6
E_3	0	0	0	v_3	0	0
F_3	0	0	v_4	0	0	0
K_3	v_1	v_2	q^2v_3	$q^{-2}v_4$	v_5	v_6

Table 2.4: The fundamental $U_q(\mathfrak{sp}_6)$ representation V

We have the decomposition

$$V \otimes V = \Gamma_{2,0} \oplus W \oplus \mathbb{C}(q)$$

where $\Gamma_{2,0,0}$ is a q-deformation of $\mathrm{Sym}^2 V$ and we are slightly abusing notation and use V, W to denote $\Gamma_{1,0,0}$ and $\Gamma_{0,1,0}$, respectively.

The irreducible representation W has dimension 14:

- (Weight $\pm\epsilon_i \pm \epsilon_j$) $w_{ij} = qv_i \otimes v_j - v_j \otimes v_i$ for $1 \leq i < j \leq 6$, $i + j \neq 7$
- (Weight 0) $w_{0,1} = v_1 \otimes v_6 - v_6 \otimes v_1 + qv_2 \otimes v_5 - q^{-1}v_5 \otimes v_2$
- (Weight 0) $w_{0,2} = v_2 \otimes v_5 - v_5 \otimes v_2 + qv_3 \otimes v_4 - q^{-1}v_4 \otimes v_3$

The action of $U_q(\mathfrak{sp}_6)$ on W is recorded below. Note that this follows by the Hopf algebra structure of $U_q(\mathfrak{sp}_6)$ on $V \otimes V$, but it may be useful to have as a quick reference.

W	w_{12}	w_{13}	w_{14}	w_{15}	w_{23}	w_{24}	w_{26}	w_{35}	w_{36}	w_{45}	w_{46}	w_{56}	$w_{0,1}$	$w_{0,2}$
E_1	0	0	0	0	w_{13}	w_{14}	$w_{0,1}$	0	w_{35}	0	w_{45}	0	$[2]w_{15}$	w_{15}
F_1	0	w_{23}	w_{24}	$w_{0,1}$	0	0	0	w_{36}	0	w_{46}	0	0	$[2]w_{26}$	w_{26}
K_1	w_{12}	qw_{13}	qw_{14}	q^2w_{15}	$q^{-1}w_{23}$	$q^{-1}w_{24}$	$q^{-2}w_{26}$	qw_{35}	$q^{-1}w_{36}$	qw_{45}	$q^{-1}w_{46}$	w_{56}	$w_{0,1}$	$w_{0,2}$
E_2	0	w_{12}	0	w_{14}	0	0	0	$w_{0,2}$	w_{26}	0	0	w_{46}	w_{24}	$[2]w_{24}$
F_2	w_{13}	0	w_{15}	0	0	$w_{0,2}$	w_{36}	0	0	0	w_{56}	0	w_{35}	$[2]w_{35}$
K_2	qw_{12}	$q^{-1}w_{13}$	qw_{14}	$q^{-1}w_{15}$	w_{23}	q^2w_{24}	qw_{26}	$q^{-2}w_{35}$	$q^{-1}w_{36}$	w_{45}	qw_{46}	$q^{-1}w_{56}$	$w_{0,1}$	$w_{0,2}$
E_3	0	0	w_{13}	0	0	w_{23}	0	0	0	w_{35}	w_{36}	0	0	0
F_3	0	w_{14}	0	0	w_{24}	0	0	w_{45}	w_{46}	0	0	0	0	0
K_3	w_{12}	q^2w_{13}	$q^{-2}w_{14}$	w_{15}	q^2w_{23}	$q^{-2}w_{24}$	w_{26}	q^2w_{35}	q^2w_{36}	$q^{-2}w_{45}$	$q^{-2}w_{46}$	w_{56}	$w_{0,1}$	$w_{0,2}$

Table 2.5: The fundamental $U_q(\mathfrak{sp}_6)$ representation W

Again, we have $\mathbb{C}(q)$ as an irreducible summand of $V \otimes V$; it is spanned by

$$z = q^3 v_1 \otimes v_6 - q^{-3} v_6 \otimes v_1 - q^2 v_2 \otimes v_5 + q^{-2} v_5 \otimes v_2 + q v_3 \otimes v_4 - q^{-1} v_4 \otimes v_3.$$

The irreducible representation $\Gamma_{2,0,0}$ is 21 dimensional, with one-dimensional weight spaces of weight $\pm 2L_i$, $\pm L_i \pm L_j$, and a 3-dimensional weight zero space, with weight vectors:

- (Weight $\pm 2L_i$) $v_i \otimes v_i$ for $1 \leq i \leq 6$
- (Weight $\pm L_i \pm L_j$) $q^{-1} v_i \otimes v_j + v_j \otimes v_i$ for $1 \leq i < j \leq 6$ with $i + j \neq 7$
- (Weight 0) $q^{-1} v_1 \otimes v_6 + q v_6 \otimes v_1 + v_2 \otimes v_5 + v_5 \otimes v_2$
- (Weight 0) $q^{-1} v_2 \otimes v_5 + q v_5 \otimes v_2 + v_3 \otimes v_4 + v_4 \otimes v_3$
- (Weight 0) $q^{-1} v_3 \otimes v_4 + q v_4 \otimes v_3$

Now we have explicitly given the decomposition of $V \otimes V = \Gamma_{2,0,0} \oplus W \oplus \mathbb{C}(q)$ into irreducible representations.

Finally, the irreducible representation $X = \Gamma_{0,0,1}$ also has dimension 14; consisting of single dimensional weight spaces of weights $\pm L_1 \pm L_2 \pm L_3$ and $\pm L_i$ for $i = 1, 2, 3$. They are (viewing $X \subset W \otimes V$),

- $x_1 = q^{-1} w_{23} \otimes v_1 - w_{13} \otimes v_2 + q w_{12} \otimes v_3$
- $x_2 = F_3.w_1 = q^{-1} w_{24} \otimes v_1 - w_{14} \otimes v_2 + q w_{12} \otimes v_4$
- $x_3 = F_2.w_2 = w_{0,2} \otimes v_1 - q w_{15} \otimes v_2 - w_{14} \otimes v_3 + q^2 w_{13} \otimes v_4 + q w_{12} \otimes v_5$
- $x_4 = F_1.w_3 = w_{26} \otimes v_1 + q^{-1} w_{0,2} \otimes v_2 - w_{0,1} \otimes v_2 - w_{24} \otimes v_3 + q^2 w_{23} \otimes v_4 + q w_{12} \otimes v_6$
- $x_5 = F_2.w_4 = w_{36} \otimes v_1 + q^{-1} w_{35} \otimes v_2 - w_{0,1} \otimes v_3 + q^2 w_{23} \otimes v_5 + q w_{13} \otimes v_6$
- $x_6 = F_1.w_5 = q^{-1} w_{36} \otimes v_2 - w_{26} \otimes v_3 + q w_{23} \otimes v_6$
- $x_7 = F_2.w_3 = q^{-1} w_{35} \otimes v_1 - w_{15} \otimes v_3 + q w_{13} \otimes v_5$
- $x_8 = F_3.w_7 = q^{-1} w_{45} \otimes v_1 - w_{15} \otimes v_4 + q w_{14} \otimes v_5$

- $x_9 = F_1.w_8 = w_{46} \otimes v_1 + q^{-1}w_{45} \otimes v_2 - w_{0,1} \otimes v_4 + q^2 w_{24} \otimes v_5 + qw_{14} \otimes v_6$
- $x_{10} = F_2.w_9 = w_{56} \otimes v_1 + q^{-1}w_{45} \otimes v_3 - qw_{35} \otimes v_4 - w_{0,1} \otimes v_5 + qw_{0,2} \otimes v_5 + qw_{15} \otimes v_6$
- $x_{11} = F_1.w_9 = q^{-1}w_{46} \otimes v_2 - w_{26} \otimes v_4 + qw_{24} \otimes v_6$
- $x_{12} = F_2.w_{11} = w_{56} \otimes v_2 + q^{-1}w_{46} \otimes v_3 - qw_{36} \otimes v_4 - w_{26} \otimes v_5 + qw_{0,2} \otimes v_6$
- $x_{13} = F_2.w_{12} = q^{-1}v_{56} \otimes x_3 - v_{36} \otimes x_5 + qv_{35} \otimes x_6$
- $x_{14} = F_3.w_{13} = q^{-1}v_{56} \otimes x_4 - v_{46} \otimes x_5 + qv_{45} \otimes x_6$

As before, we define certain multiples of the projection and inclusion maps $i_{\mathbb{C}(q)} : \mathbb{C}(q) \to V \otimes V$, $i_W : W \to V \otimes V$, $p_{\mathbb{C}(q)} : V \otimes V \to \mathbb{C}(q)$, $p_W : V \otimes V \to W$. The inclusion maps $i_{\mathbb{C}(q)}$, i_W are defined above by writing how $\mathbb{C}(q), W$ sit inside of $V \otimes V$. The maps $p_{\mathbb{C}(q)}$, p_W are defined by:

$$p_{\mathbb{C}(q)} : \begin{cases} v_i \otimes v_j \mapsto 0 \text{ if } i+j \neq 7 \\ v_1 \otimes v_6 \mapsto -q^3 \\ v_2 \otimes v_5 \mapsto q^2 \\ v_3 \otimes v_4 \mapsto -q \\ v_4 \otimes v_3 \mapsto q^{-1} \\ v_5 \otimes v_2 \mapsto -q^{-2} \\ v_6 \otimes v_1 \mapsto q^{-3} \end{cases} \quad \text{and} \quad p_W : \begin{cases} v_i \otimes v_i \mapsto 0 \\ v_i \otimes v_j \mapsto -[3]w_{ij} \text{ if } i+j \neq 7, i<j \\ v_j \otimes v_i \mapsto q^{-1}[3]w_{ij} \text{ if } i+j \neq 7, i<j \\ v_1 \otimes v_6 \mapsto -[2]w_{0,1} + w_{0,2} \\ v_2 \otimes v_5 \mapsto -q^2 w_{0,1} - q^{-1}w_{0,2} \\ v_3 \otimes v_4 \mapsto qw_{0,1} - q[2]w_{0,2} \\ v_4 \otimes v_3 \mapsto -q^{-1}w_{0,1} + q^{-1}[2]w_{0,2} \\ v_5 \otimes v_2 \mapsto q^{-2}w_{0,1} + qw_{0,2} \\ v_6 \otimes v_1 \mapsto [2]w_{0,1} - w_{0,2} \end{cases}$$

Using these maps, we compute

$$\begin{aligned} p_{\mathbb{C}(q)} \circ i_{\mathbb{C}(q)} &= p_{\mathbb{C}(q)}(q^3 v_1 \otimes v_6 - q^{-3}v_6 \otimes v_1 - q^2 v_2 \otimes v_5 + q^{-2}v_5 \otimes v_2 + qv_3 \otimes v_4 - q^{-1}v_4 \otimes v_3) \\ &= q^3(-q^{-3}) - q^{-3}(q^{-3}) - q^2(q^2) + q^{-2}(-q^{-2}) + q(-q) - q^{-1}(q^{-1}) \\ &= -q^{-6} - q^{-4} - q^{-2} - q^2 - q^4 - q^6 \\ &= -\frac{[3][8]}{[4]} \end{aligned}$$

and

$$\begin{aligned}(p_W \circ i_W)(w_{12}) &= p_W(qv_1 \otimes v_2 - v_2 \otimes v_1)\\ &= q(-[3]w_{12}) - (q^{-1}[3]w_{12})\\ &= -[2][3]w_{12}\end{aligned}$$

so that $p_{\mathbb{C}(q)} \circ i_{\mathbb{C}(q)} = -\frac{[3][8]}{[4]} id_{\mathbb{C}(q)}$ and $p_W \circ i_W = id_W$.

Similar to the previous section, we compute that $(p_{\mathbb{C}(q)} \otimes id_V) \circ (id_V \otimes i_{\mathbb{C}(q)}) = id_V$ and $(id_V \otimes p_{\mathbb{C}(q)}) \circ (i_{\mathbb{C}(q)} \circ id_V) = id_V$.

We may use these maps to define maps $i_{\mathbb{C}(q)\to W\otimes W} : \mathbb{C}(q) \to W \otimes W$ and $p_W : W \otimes W \to \mathbb{C}(q)$. We make the definitions

$$i_{\mathbb{C}(q)\to W\otimes W} = -\frac{1}{[2][3]}(p_W \otimes p_W) \circ (id_V \otimes i_{\mathbb{C}(q)} \otimes id_V) \circ (id_V \otimes p_{\mathbb{C}(q)} \otimes q_V) \circ (i_{\mathbb{C}(q)} \otimes i_{\mathbb{C}(q)})$$

and

$$p_{W\otimes W\to\mathbb{C}(q)} = -\frac{1}{[2][3]}(p_{\mathbb{C}(q)} \otimes p_{\mathbb{C}(q)}) \circ (id_V \otimes i_{\mathbb{C}(q)} \otimes id_V) \circ (id_V \otimes p_{\mathbb{C}(q)} \otimes id_V) \otimes (i_W \otimes i_W).$$

A computation[3] then shows that $p_{W\otimes W\to\mathbb{C}(q)} \circ i_{\mathbb{C}(q)\to W\otimes W} = \frac{[7][8]}{[4]} id_{\mathbb{C}(q)}$.

We also have maps $i_X : X \to W \otimes V$, $j_X : X \to V \otimes W$, $p_X : W \otimes V \to X$, $q_X : V \otimes W \to X$. The map i_X is defined by specifying how X sits inside $W \otimes V$; for example,

$$i_X(x_1) = q^{-1}w_{23} \otimes v_1 - w_{13} \otimes v_2 + qw_{12} \otimes v_3.$$

Including $X \hookrightarrow V \otimes W$ rather than $X \subset W \otimes V$, the tensor factors are flipped and q is replaced by q^{-1}. For example, in $W \otimes V$, the highest weight vector x_1 is instead given by

$$j_X(x_1) = qv_1 \otimes w_{23} - v_2 \otimes w_{13} + q^{-1}v_3 \otimes w_{12}.$$

[3]We omit this computation for its length; for example, in the process, we have a expression of 36 terms!

We define j_X accordingly, by being the image of i_X with tensor factors switched and q replaced by q^{-1}.

The domains of p_X, q_X have dimension $6 \cdot 14 = 84$, so we do not record the image of every element of them here. Rather, we will find a much better way to record these maps, utilizing the fact that $U_q(\mathfrak{sp}_4)$ is a subalgebra of $U_q(\mathfrak{sp}_6)$. However, for the sake of illustration, we do define the maps p_X, q_X on several elements:

$$p_X : \begin{cases} w_{23} \otimes v_1 \mapsto -q^{-1}[3]x_1 \\ w_{13} \otimes v_2 \mapsto [3]x_1 \\ w_{12} \otimes v_3 \mapsto -q[3]x_1 \end{cases} \quad \text{and} \quad q_X : \begin{cases} v_1 \otimes w_{23} \mapsto -q[3]x_1 \\ v_2 \otimes w_{13} \mapsto [3]x_1 \\ v_3 \otimes w_{12} \mapsto -q^{-1}[3]x_1 \end{cases}$$

One can check that p_X, q_X are $-[3]^2$ times the projection maps.

2.4 Monoidal Categories

Next, let us discuss monoidal categories and their diagrammatic language. We follow the excellent paper [31].

Definition 2.4.1. A *category* $\mathcal{C}$ is a class of objects $ob(\mathcal{C})$ along with, for each pair of objects X, Y, a class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (we sometimes omit the subscript if $\mathcal{C}$ is understood). The elements of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are called *morphisms* (from X to Y). If $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we also write $f : X \to Y$. Morphisms have the additional structure:

- A function $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ called *composition*; we write $g \circ f$ for the image of (f, g).

- For every object X, a morphism $id_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that, for any morphism $u \in \mathrm{Hom}(X, Y)$ and any morphism $v \in \mathrm{Hom}(Z, X)$, we have $u \circ id_X = u$ and $id_X \circ v = v$.

- The composition of morphisms is associative (whenever defined).

A morphism $f : A \to B$ is an *isomorphism* if it has a two-sided inverse; i.e. there is a morphism $g : B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$.

A *functor* F from a category $\mathcal{C}_1$ to $\mathcal{C}_2$ consists of a map sending each object $X \in ob(\mathcal{C}_1)$ to an object $F(X) \in ob(\mathcal{C}_2)$ and, for all objects X, Y of $\mathcal{C}_1$, a set map $F : \mathrm{Hom}_{\mathcal{C}_1}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_2}(F(X), F(Y))$ such that

- $F(id_X) = id_{F(X)}$.
- $F(f \circ g) = F(f) \circ F(g)$.

The functor F is called *full* if for any two objects x, y of $\mathcal{C}_1$, the induced map $\mathrm{Hom}_{\mathcal{C}_1}(X,Y) \to \mathrm{Hom}_{\mathcal{C}_2}(F(X), F(Y))$ is surjective; F is called *faithful* if for any two objects x, y of $\mathcal{C}_1$, the induced map $\mathrm{Hom}_{\mathcal{C}_1}(X,Y) \to \mathrm{Hom}_{\mathcal{C}_2}(F(X), F(Y))$ is injective. If F is full and faithful, we say F is fully faithful. Finally, F is called *essentially surjective* if each object of $\mathcal{C}_2$ is isomorphic to an object of the form $F(x)$ for some object x of $\mathcal{C}F_1$. The functor F is an *equivalence of categories* if it is fullly faithful and essentailly surjective.

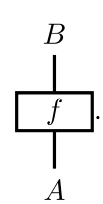
Given categories $\mathcal{C}, \mathcal{D}$ and functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\alpha : F \Rightarrow G$ assigns to every object $X \in ob(\mathcal{C})$ a morphism $\alpha_X : F(X) \to G(X)$ such that for every $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, we have that

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

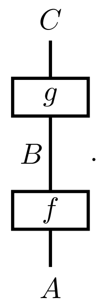
commutes. Finally, a *natural isomorphism* between functors F and G is a natural transformation from F to G with a two-sided inverse.

Throughout this section, we will use the category of finite dimensional $\Bbbk$-vector spaces as a motivating example. The category $\mathrm{Vect}_\Bbbk$ has objects all finite dimensional $\Bbbk$-vector spaces and morphisms $\mathrm{Hom}_{\mathrm{Vect}_\Bbbk}(V, W)$ the collection of all linear transformations from V to W. Another example of a category is the category Set, which has objects sets and $\mathrm{Hom}_{\mathrm{Set}}(X,Y)$ the set of all functions from X to Y. There is a functor $F : \mathrm{Vect}_\Bbbk \to \mathrm{Set}$, which, on objects, forgets about the vector space structure, and on morphisms, views a linear transformation as a function between sets.

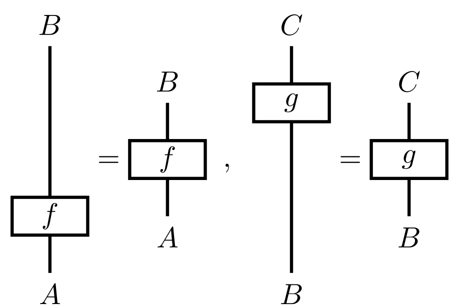
Let us now discuss a diagrammatic formalism for categories. We depict an object by a label, and a morphism from one object to another by drawing a strand between labels with a box containing a morphism. We shall always read from bottom to top. For example, for objects A, B, we depict a morphism $f : A \to B$ by



Composition is given by vertical stacking, and identity morphisms are shown as a strand with no box. For example, given morphisms $f : A \to B$ and $g : B \to C$, we draw $g \circ f$ as



The fact that function composition is associative makes this notation (with three or more boxes) well-defined. Further, note that the axioms that $id_B \circ f = f$ and $f \circ id_A = f$ translate to the diagrams



respectively, which hold up to isotopy in the graphical language.

The category $\mathrm{Vect}_{\Bbbk}$ has more structure than we've described so far. For example, given two objects V, W of $\mathrm{Vect}_{\Bbbk}$, there is another object $V \otimes W$ of $\mathrm{Vect}_{\Bbbk}$. Further, given morphisms $S : V \to X$ and $T : W \to Y$, there is a corresponding morphism $S \otimes T : V \otimes W \to X \otimes Y$. This is the structure we will formalize now.

Definition 2.4.2. A *monoidal* category is a category equipped with the following:

- A functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- A (monoidal) unit I
- For every pair of morphisms $f : A \to B$ and $g : C \to D$, a morphism $f \otimes g : A \otimes C \to B \otimes D$
- A natural isomorphism $\alpha : ((-) \otimes (-)) \otimes (-) \to (-) \otimes ((-) \otimes (-))$, where the components $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ are called the *associator*

- A natural isomorphism $\lambda : I \otimes (-) \to (-)$ with components $\lambda_X : I \otimes X \to X$
- A natural isomorphism $\rho : (-) \otimes I \to (-)$ with components $\rho_X : X \otimes I \to X$

such that the diagrams commute:

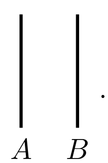
- The triangle identity:

$$\begin{array}{ccc} (X \otimes 1) \otimes Y & \xrightarrow{\alpha_{X,1,Y}} & X \otimes (1 \otimes Y) \\ & {\scriptstyle \rho_X \otimes id_Y} \searrow \quad \swarrow {\scriptstyle id_X \otimes \lambda_Y} & \\ & X \otimes Y & \end{array}$$

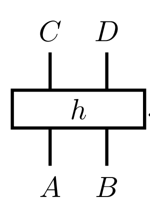
- The pentagon identity:

$$\begin{array}{ccc} & (W \otimes X) \otimes (Y \otimes Z) & \\ {\scriptstyle \alpha_{W \otimes X,Y,Z}} \nearrow & & \searrow {\scriptstyle \alpha_{W,X,Y \otimes Z}} \\ ((W \otimes X) \otimes Y) \otimes Z & & W \otimes (X \otimes (Y \otimes Z)) \\ \downarrow {\scriptstyle \alpha_{W,X,Y} \otimes id_Z} & & \uparrow {\scriptstyle id_W \otimes \alpha_{X,Y,Z}} \\ (W \otimes (X \otimes Y)) \otimes Z & \xrightarrow{\alpha_{W,X \otimes Y,Z}} & W \otimes ((X \otimes Y) \otimes Z) \end{array}$$

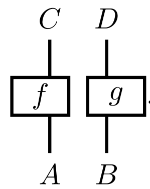
Graphically, we depict the unit object I as an empty label (i.e., we draw nothing for I). The operation $\otimes$ is given by horizontal juxtoposition. For example, we draw (the identity morphism of) $A \otimes B$ as



We write a morphism $h : A \otimes B \to C \otimes D$ as



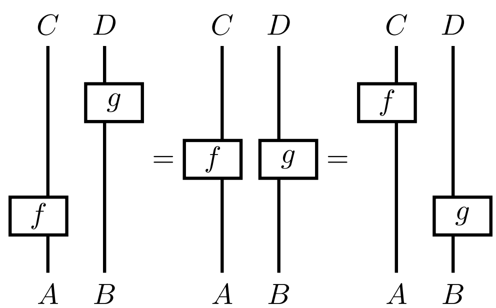
If $f : A \to C$ and $g : B \to D$ are morphisms, their tensor product is drawn



Again, equalities between morphisms in a monoidal category correspond exactly to diagrams up to isotopy. For example, for morphisms $f : A \to C$ and $g : B \to D$, the identity

$$(id_C \otimes g) \circ (f \otimes id_B) = f \otimes g = (f \otimes id_D) \circ (id_A \otimes g)$$

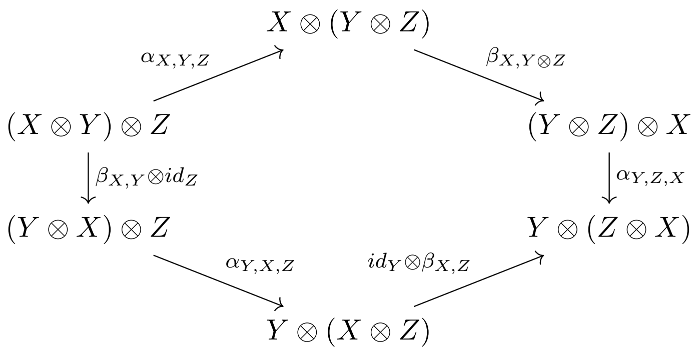
corresponds to the isotopy



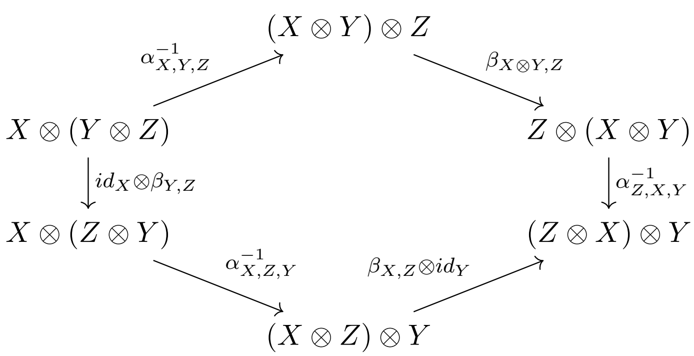
in the graphical language.

Returning to the example of $\mathrm{Vect}_{\mathbb{k}}$, we actually have that $V \otimes W \cong W \otimes V$ for all vector spaces V, W. This is the structure we shall formalize next.

Definition 2.4.3. A *braided* monoidal category is a monoidal category along with a natural isomorphism $\beta_{A,B} : A \otimes B \to B \otimes A$ (called a braiding) for each pair of objects A, B. The braiding is compatible with the associators, in the sense that the two diagrams

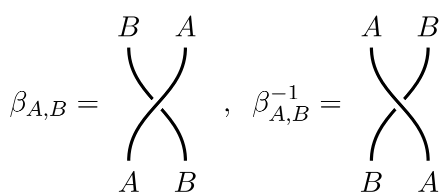


and

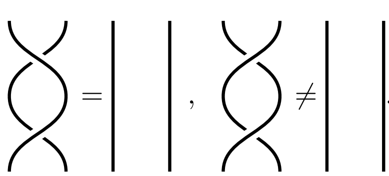


commute, called the hexagon relations.

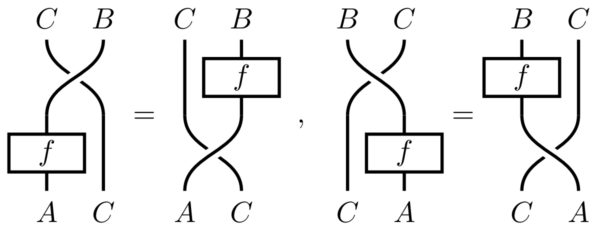
Graphically, we write the isomorphism $\beta_{A,B}$ and its inverse as



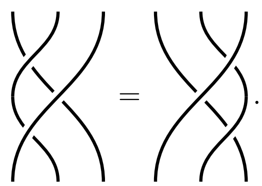
In general, $\beta \circ \beta \neq id$ (if this equality does hold, we call the category *symmetric*. The fact that $\beta^{-1} \circ \beta = id$ and $\beta \circ \beta \neq id$ are apparent in the graphical language:



The naturality of β says that for all morphisms $f : A \to B$, we have



One consequence of the naturality of β along with the hexagon relation is the *Yang-Baxter equation*



In our example of $\mathrm{Vect}_{\Bbbk}$ the braiding $\beta_{V,W}$ is given by $v \otimes w \mapsto w \otimes v$ (and extending linearly). Since $\beta_{Y,X} \circ \beta_{X,Y} = id_{X \otimes Y}$, we actually have that $\mathrm{Vect}_{\Bbbk}$ is symmetric. The categories of interest in this work, however, are not symmetric. Again, $\mathrm{Vect}_{\Bbbk}$ has more structure still; Given a vector space V, there is a *dual* vector space V^*. The notion of dual objects is the next concept we wish to formalize.

Definition 2.4.4. An exact pairing between objects A and B is a pair of morphisms $\eta : I \to B \otimes A$ and $\epsilon : A \otimes B \to I$ such that the following triangles commute:

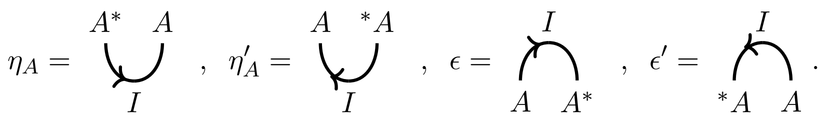
$$\begin{array}{ccc} A & \xrightarrow{id_A \otimes \eta} & A \otimes B \otimes A \\ & \searrow^{id_A} & \downarrow{\scriptstyle \epsilon \otimes id_A} \\ & & A \end{array} \quad , \quad \begin{array}{ccc} B & \xrightarrow{\eta \otimes id_B} & B \otimes A \otimes B \\ & \searrow^{id_B} & \downarrow{\scriptstyle id_B \otimes \epsilon} \\ & & B \end{array}$$

In this case, B is called the *right dual* of A and A is called the *left dual* of B.

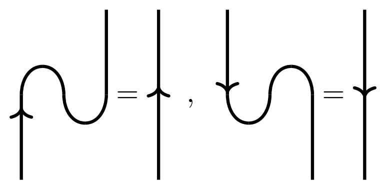
Finally, a category is called *(left/right) autonomous* if every object A has a (left/right) dual object (denoted A^* for right dual or ${}^*A$ for left dual).

Graphically, we depict A and its duals both with a strand labelled by A, but with an orientation upwards for A and downwards for A^* and ${}^*A$. We write the maps $\eta_A : I \to A^* \otimes A$, $\eta'_A : I \to A \otimes {}^*A$,

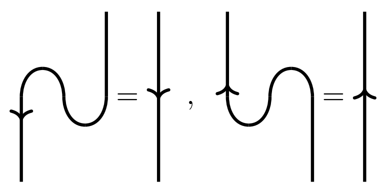
$\epsilon_A : A \otimes A^* \to I$, and $\epsilon'_A : {}^*A \otimes A \to I$ as follows:



The properties that the morphisms of the exact pairing must satisfy are then the "snake relations"

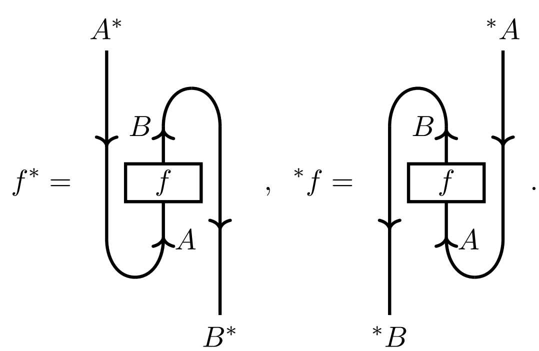


if the category is right autonomous, and



if the category is left autonomous.

Given a morphism $f : A \to B$, may define new morphisms $f^* : B^* \to A^*$ and ${}^*f : {}^*B \to {}^*A$, called the *adjoint mates* of f as



In the case of an autonomous category which is also braided, we have the following.

Lemma 2.4.5. A braided monoidal category is autonomous if and only if it is right autonomous.

Proof. As in [31] Let $\eta : I \to B \otimes A$ and $\epsilon : A \otimes B \to I$ be an exact pairing. Then we have that $\beta^{-1}_{A,B} \circ \eta : I \to A \otimes B$ and $\epsilon \circ \beta_{B,A}$ also are exact pairings. So if B is a right dual of A (via the first maps), it is also a left dual of A (via the second maps). $\square$

Returning to our example, we see that $\mathrm{Vect}_{\Bbbk}$ is autonomous. An object V has dual $V^* = \mathrm{Hom}(V, \Bbbk)$, and the pairings are defined by $\epsilon_A(v \otimes f) = f(v)$ and $\eta_A : 1 \mapsto \sum v_i^* \otimes v_i$ where $\{v_i\}$ is a basis of V. The maps ϵ'_A and η'_A are similar. Checking the snake relations are then straightforward. Given a morphism $T : V \to W$, the adjoint mate $T^* : W^* \to V^*$ is given by $T^*(f) = f \circ T$. In the case of $\mathrm{Vect}_{\Bbbk}$, there is a natural isomorphism between a vector space V and $(V^*)^*$ (given by $v \mapsto (f \mapsto f(v))$). This is the next concept we would like to formalize.

Definition 2.4.6. A *pivotal* category is a (right) autonomous category along with a monoidal natural isomorphism $i_A : A \to (A^*)^*$ such that

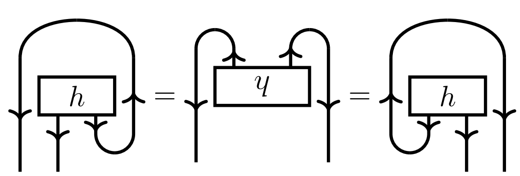
$$\begin{array}{ccc} A^* & \xrightarrow{i_{A^*}} & A^{***} \\ & \searrow^{id_{A^*}} & \downarrow{\scriptstyle i_A^*} \\ & & A^* \end{array}$$

commutes.

A pivotal category is always autonomous (right duals are immediately also left duals). The assumption that i_A is a monoidal natural transformation means that i_I is cannonoical and the following commutes:

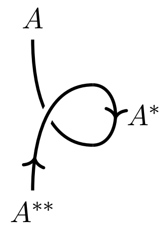
$$\begin{array}{ccccc} & & A \otimes B & & \\ & \swarrow^{i_A \otimes i_B} & & \searrow^{i_{A \otimes B}} & \\ A^{**} \otimes B^{**} & & \xrightarrow{\quad\cong\quad} & & (A \otimes B)^{**} \end{array}$$

The diagrammatics for a pivotal category are the same as for an autonomous category, but we have more freedom in the types of diagrams we can draw. For example, if $h : A^* \otimes A^* \to I$ is a morphism, we have the following equalities:

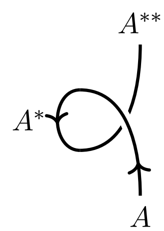


where the morphism in the middle was rotated.

In a braided autonomous category, there is a natural isomorphism $b_A : A^{**} \to A$ given by

$b_A =$ 

with inverse

$b_A^{-1} =$  .

Since b is not a monoidal natural transformation, this need not define a pivotal structure. However, in one special case, it does. First, a definition.

Definition 2.4.7. A *twist* on a braided monoidal category is a natural family of isomorphisms $\theta_A : A \to A$ such that $\theta_I = id_I$ and, for all A, B,

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\beta_{A,B}} & B \otimes A \\ \downarrow{\scriptstyle \theta_{A\otimes B}} & & \downarrow{\scriptstyle \theta_B \otimes \theta_A} \\ A \otimes B & \xleftarrow[\beta_{B,A}]{} & B \otimes A \end{array}$$

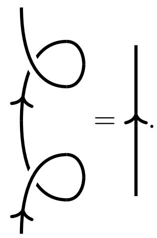
commutes. If a braided monoidal category has a twist, we say it is a *balanced* monoidal category.

We then have the connection, a proof of which may be found in [31].

Lemma 2.4.8. Given a braided autonomous category $\mathcal{C}$, if there is a twist θ, then $i_A = b_A^{-1} \circ \theta_A$ defines a pivotal structure. Conversely, given a pivotal structure i, then $\theta_A = b_A \circ i_A$ defines a twist.

Finally, a pivotal braided category (or balanced and autonomous) is said to be *ribbon* (or *tortile*) if we also have one of the following (equivalent) identities

1.



2. $\theta_{A^*} = (\theta_A)^*$

When graphically describing a ribbon category, we may replace the strands with ribbons. The twist map is then depicted as a twist as below in [31].

Figure 2.1: The twist map θ_A in a ribbon category

If an object A is isomorphic to A^*, a *coherent self-duality* is an isomorphism $h_A : A \to A^*$ such that the diagram

$$\begin{array}{ccc} A^* & \xrightarrow{i_{A^*}} & (A^*)^* \\ & \searrow^{\theta_{A^*}} & \downarrow h_A^* \\ & & A^* \end{array}$$

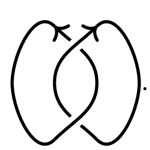
commutes.

We are now prepared to more precisely state the connection between link invariants and quantum groups. The following theorem is due to Reshetikhin-Turaev in [26].

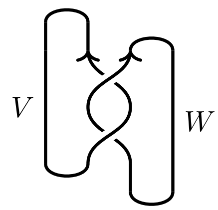
Theorem 2.4.9. For $\mathfrak{g}$ semisimple, $\mathbf{Rep}(U_q(\mathfrak{g}))$ is a *ribbon* category.

Thus, given a link diagram $\mathcal{L}_{\mathcal{D}}$, we may label each component of $\mathcal{L}$ by an irreducible $U_q(\mathfrak{g})$ representation, and interpret the link diagram as an endomorphism of the trivial representation; that is, as an element of $\mathbb{C}(q)$.

For example, earlier we saw the diagram of an oriented Hopf link



If we color each of the components by an irreducible $U_q(\mathfrak{g})$ representation, the diagram (which we have drawn slightly differently)



may be interpreted as the morphism

$$\epsilon'_V \circ (id_{V^* \otimes V} \otimes \epsilon_W) \circ (id_{V^*} \otimes \beta_{W,V} \otimes id_{W^*}) \circ (id_{V^*} \otimes \beta_{V,W} \otimes id_{W^*}) \circ (\eta_V \otimes id_{W \otimes W^*}) \circ \eta^*_W$$

from $\mathbb{C}(q)$ to itself; thus, as an element of $\mathbb{C}(q)$.

2.5 Temperly-Lieb

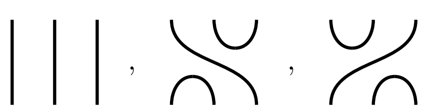
In this section, we discuss the Temperly-Lieb category, which is the prototypical diagrammatic presentation of a category of representations. We start with the more familiar Temperly-Lieb algebra.

Definition 2.5.1. We define the *Temperly-Lieb algebra* on n strands ($n > 0$). Let $\mathbf{TL}_n$ be the $\mathbb{C}(q)$-algebra generated by $1, e_1, \ldots, e_{n-1}$ modulo relations:

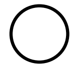
- $e_i e_j = e_j e_i$ if $|i - j| > 1$
- $e_i e_{i\pm1} e_i = e_i$
- $e_i^2 = -[2]e_i$.

We claim $\mathbf{TL}_n$ admits a diagrammatic presentation. Take the square $I \times I$, and mark n points (called *boundary points*) on $I \times \{0\}$ and on $I \times \{1\}$. Define a *crossingless n, n tangle* to be an isotopy class of smoothly embedded disjoint 1-manifolds, those with boundary having boundary points exactly the marked points of the marked square.

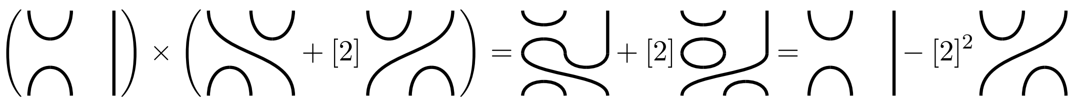
For example, when $n = 3$,



are all crossingless $3, 3$ tangles. Now, let T_n be the $\mathbb{C}(q)$-vector space of formal $\mathbb{C}(q)$-linear sums of crossingless n, n tangles modulo the relation

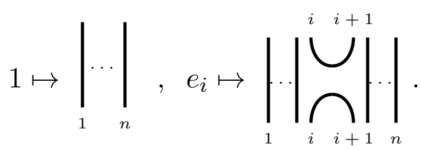
 $= -[2]$.

Endow T_n with the structure of an algebra by stacking and re-scaling in the y-direction. For example,

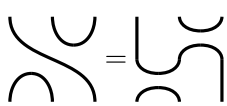


Theorem 2.5.2. The algebra T_n is a diagrammatic presentation of $\mathbf{TL}_n$.

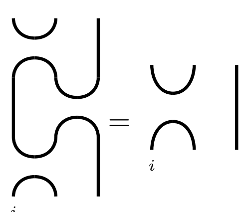
Proof. Define a map $\phi : \mathbf{TL}_n \to T_n$ by



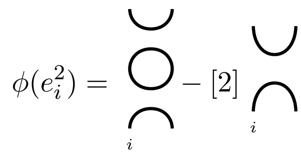
First, we claim that ϕ is surjective. We only give the idea here; a full proof is in [35]. First, given a crossingless tangle, apply an isotopy to write it in terms only of vertical segments and semicircles (with horizontal diameter). This is possible by an argument involving the number of turns an arc makes when it connects marked points on the same side or opposite side. Finally, arcs on opposite sides until the crossingless tangle is in the desired form. As an easy example, consider

 $= \phi(e_2 e_1)$.

Let's verify that ϕ is injective. First, if $|i - j| > 1$, we have that $\phi(e_i e_j) = \phi(e_j e_i)$ by isotopy. We also have

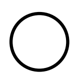
$\phi(e_i e_{i+1} e_i) =$  $= e_i$

where we have omitted drawing the other strands. Similarly, $\phi(e_i e_{i-1} e_i) = \phi(e_i)$. Finally,



so ϕ is injective. $\square$

In fact, we can "upgrade" the Temperly-Lieb algebra to a category. Define the *Temperly-Lieb category* **TL** to be the $\mathbb{C}(q)$-linear monoidal category with objects $\mathbb{Z}_{\geq 0}$ and $\mathrm{Hom}(m, n)$ the set of formal $\mathbb{C}(q)$-linear combination of crossingless m, n tangles (an isotopy class of smoothly embedded disjoint copies of I with boundary points on m marked points on $I \times \{0\}$ and n marked points on $I \times \{1\}$) modulo the relation

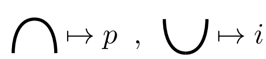


$$= -[2].$$

The monoidal structure on objects is defined by $m \otimes n = m + n$ and on morphisms by horizontal concatenation (and re-scaling in the x-direction). Morphism composition (when defined) is given by vertical concatenation (and re-scaling in the y-direction). For each $n > 0$, note that $\mathbf{TL}_n = \mathrm{End}_{\mathbf{TL}}(n)$.

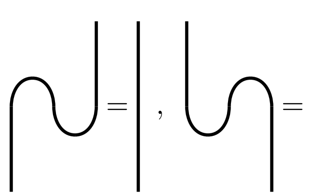
Theorem 2.5.3. There is a fully faithful, essentially surjective monoidal functor $\Psi : TL \to$ $\mathbf{FundRep}(U_q(\mathfrak{sl}_2))$.

Proof. The functor Ψ, on objects, sends $n \mapsto V^{\otimes n}$, and on morphisms, sends



where these are the projection and inclusion maps defined earlier.

It is straightforward to check that this map factors through the **TL** relations. By some Cerf and Morse theory, the only isotopy relations needed to be checked are the "snake" relations



so we must check that $(p \otimes id_V) \circ (id_V \otimes i) = id_V$ and $(id_V \otimes p) \circ (i \circ id_V) = id_V$. Indeed,

$$(p \otimes id_V) \circ (id_V \otimes i)(v_+) = qv_+ \otimes v_+ \otimes v_- - v_+ \otimes v_- \otimes v_+ = 0 - (-1)v_+ = v_+$$

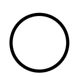
(by Schur's lemma, the snake relation holds up to a scalar, so we only need to verify it on one element). Similarly, one can check that $(id_V \otimes p) \circ (i \otimes id_V) = id_V$. Finally, we must check that $p \circ i = -[2]1_{\Bbbk}$. Indeed,

$$(p \circ i)(z) = p(qv_+ \otimes v_- - v_- \otimes v_+) = -q - q^{-1} = -[2].$$

Thus, Ψ factors through the relations of **TL**. □

Remark 2.5.4. The functor Ψ above is also essentially surjective, full, faithful, and braided. Proofs of these can be found in [29] [17].

Remark 2.5.5. Another way to describe **TL** is as the pivotal category freely generated by one self-dual object modulo the local relation

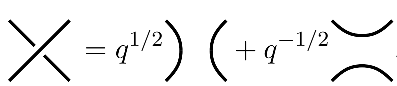

$= -[2].$

By "freely generated," we mean that we allow all finite tensor products of the object, and all compositions of tensor products of the identity and unit/counit morphisms.

By "self-dual," we mean that there is a coherent self-duality from this object to its dual.

By "local relation," we mean modulo the ideal generated (by allowing all tensor products and all compositions) of the relation.

After extending scalars to $\mathbb{C}(q^{-1/2}, q^{1/2})$, the braiding on **TL** is given by

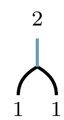


In fact, this gives the Jones polynomial, as defined in Section 2.1. As before, the fractional powers of q are a red herring; for any link $\mathcal{L}$, $V_q(\mathcal{L}) \in \mathbb{Z}[q, q^{-1}]$.

2.6 Web categories

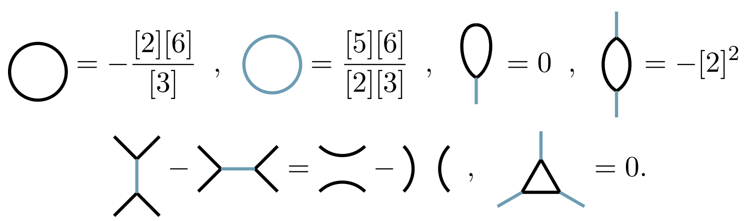
In pioneering work [17], Kuperberg extended the Temperley-Lieb description of $\mathbf{Rep}(U_q(\mathfrak{sl}_2))$ to $U_q(\mathfrak{g})$ for rank two simple $\mathfrak{g}$, $\mathfrak{sl}_2$, $\mathfrak{sp}_4$, and $\mathfrak{g}_2$. We first highlight the case $\mathfrak{g} = \mathfrak{sp}_4$, as it is the most relevant to our present work.

The type C_2 spider, denoted here by $\mathbf{Web}(\mathfrak{sp}_4)$, is a combinatorial description of the category $\mathbf{Rep}(U_q(\mathfrak{sp}_4))$. Explicitly, $\mathbf{Web}(\mathfrak{sp}_4)$ is obtained from the $\mathbb{C}(q)$-linear ribbon category pivotally generated by self-dual objects $\{1, 2\}$ and the morphism



$\in \mathrm{Hom}_{\mathbf{Web}(\mathfrak{sp}_4)}(1 \otimes 1, 2)$

with local relations:



Following [17], we refer to these graphs as *webs*. Throughout, we'll follow the conventions that we won't label the (co)domain in our morphisms, electing instead to color our web edges. Throughout, **black** denotes 1-labeled edges and blue denotes 2-labeled edges. Further, as can be inferred from the above, we read all webs as mapping from bottom to top.

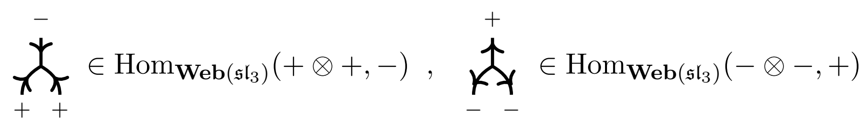
Kuperberg then proves the following result.

Theorem 2.6.1. There is an equivalence of monoidal categories $\mathbf{Web}(\mathfrak{sp}_4) \cong \mathbf{FundRep}(U_q(\mathfrak{sp}_4))$.

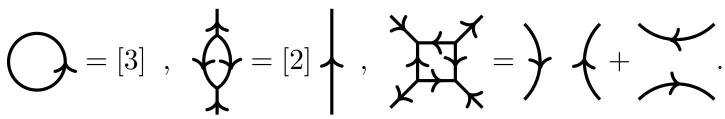
In fact, Kuperberg proves analogues of this result for all rank 2 Lie algebras (i.e. additionally for $\mathfrak{sl}_3$ and $\mathfrak{g}_2$), and extends this result to an equivalence of ribbon categories by giving explicitly formulae for the braidings in the web categories.

For the sake of completeness, we include Kuperberg's other web categories here.

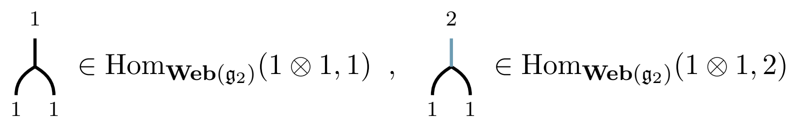
The $\mathbf{Web}(\mathfrak{sl}_3)$ category is the $\mathbb{k}$-linear ribbon category pivotally generated by objects $\{+, -\}$. These objects are not self-dual; rather, the duality structure is given by reversing the order and switching all $+$ with $-$ (and visa-versa). For example, $(+ \otimes + \otimes - \otimes +)^* = (- \otimes + \otimes - \otimes -)$. The category $\mathbf{Web}(\mathfrak{sl}_3)$ has morphisms generated by trivalent vertices



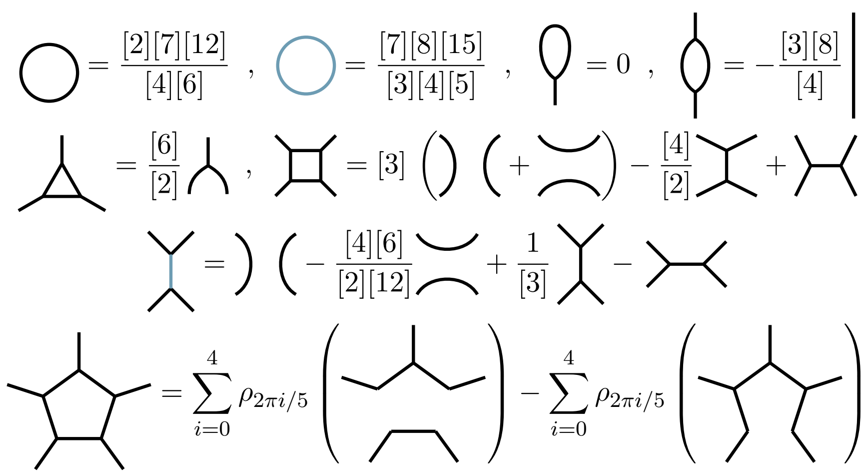
modulo the local relations



Finally, the category $\mathbf{Web}(\mathfrak{g}_2)$ is the $\Bbbk$-linear ribbon category pivotally generated by self-dual objects $\{1, 2\}$ and morphisms



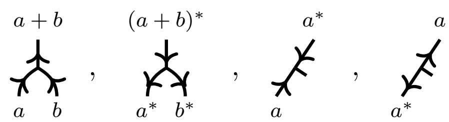
modulo the relations



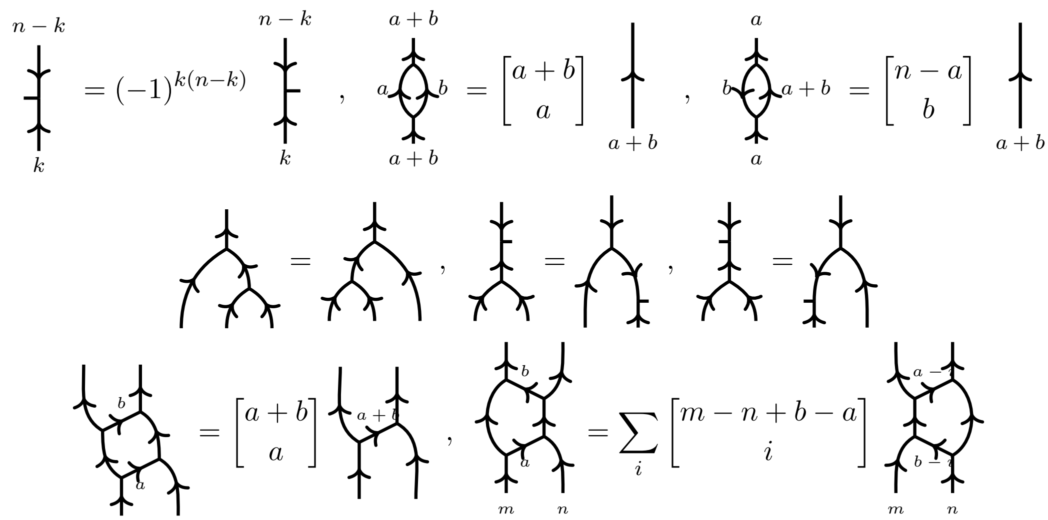
where in the last equation above, ρ_θ rotates the picture by an angle of θ (so we have 10 total terms in the last equation).

It remained an open problem for over ten years to extend Kuperberg's results beyond rank two, despite progress in the PhD theses of Kim [15] and Morrison [19]. In [4], breakthrough work of Cautis-Kamnitzer-Morrison proved the analogue of Theorem 2.6.1 for all type A simple Lie algebras, introducing skew Howe duality as a tool in studying web categories.

Again, for the sake of completeness, we include the definition of the general $\mathfrak{sl}_n$ web category. The type A_{n-1} spider, denoted $\mathbf{Web}(\mathfrak{sl}_n)$, is pivotally generated by the $2n-2$ objects $\{1, 1^*, \ldots, n-1, (n-1)^*\}$ and morphisms



Generally we will omit the $*$ label from an object and instead look at the orientation of the edge. We quotient by the relations:



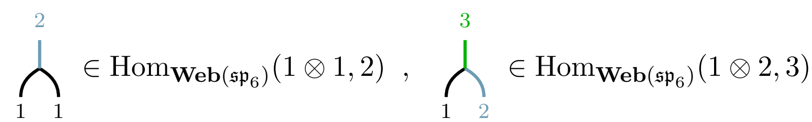
where edges which are not labelled may carry any valid labelling.

CHAPTER 3

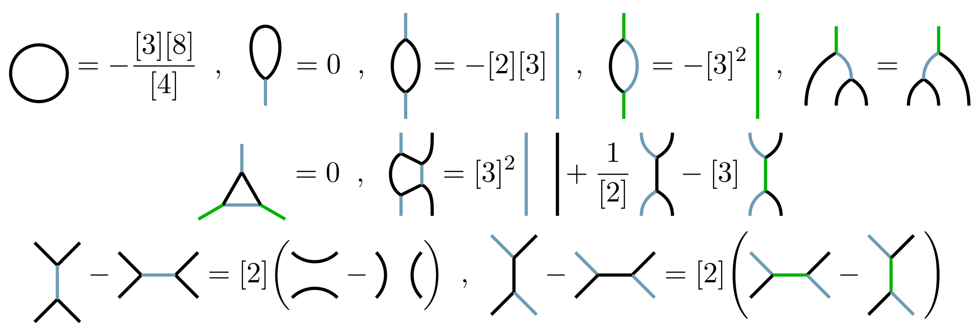
$\Psi : \mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{FundRep}(U_q(\mathfrak{sp}_6))$ is well-defined

First, we recall the definition of $\mathbf{Web}(\mathfrak{sp}_6)$

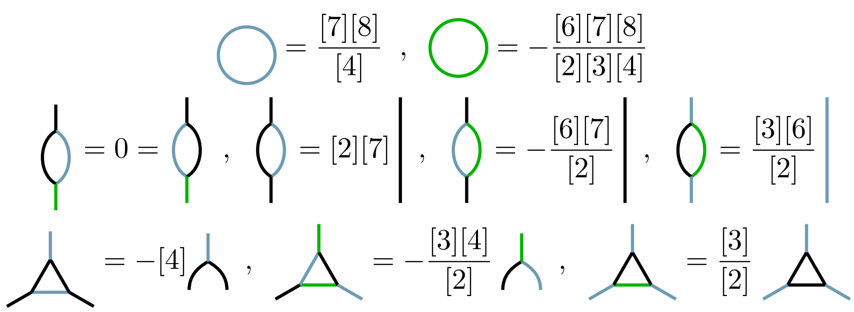
Definition 3.0.1. The category $\mathbf{Web}(\mathfrak{sp}_6)$ is the (strictly pivotal) $\mathbb{C}(q)$-linear category generated by the self-dual objects $\{1, 2, 3\}$ and with morphisms generated by



modulo the relations:



Remark 3.0.2. We record the following useful consequences of the above:



Remark 3.0.3. We remark that one is free to re-scale each trivalent vertex, which may cause the relations to look different. For example, in [28], the generating morphisms are re-scaled by $\sqrt{-1}$ and $\sqrt{-[3]/[2]}$, respectively.

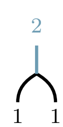
In this section, we prove that $\Psi : \mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{FundRep}(U_q(\mathfrak{sp}_6))$ is well-defined. The functor Ψ is defined on objects by sending an edge labelled by i to the irreducible $U_q(\mathfrak{sp}_6)$ representation of highest weight ω_i, and on morphisms, sending trivalent vertecies to the maps of $U_q(\mathfrak{sp}_6)$ modules.

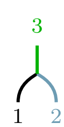
Remark 3.0.4. There is a standard choice of ribbon element for $U_q(\mathfrak{g})$, which acts on an irreducible representation with highest weight λ by K_ρ (i.e. $q^{-(\lambda,\lambda+2\rho)}$), where ρ is the half-sum of positive roots. However, following [32] and [34], we use a **non-standard** ribbon element, which instead acts by $-K_\rho$ (i.e. $-q^{-(\lambda,\lambda+2\rho)}$) on irreducible representations in an odd tensor product of V, and K_ρ $q^{-(\lambda,\lambda+2\rho)}$ on irreducible representations in an even rtensor product of V (this is well-defined due to the plethysm of type C).

One strategy is to present $\mathbf{Web}(\mathfrak{sp}_6)$ by generators and relations *as a monoidal category*, define the functor on the generating morphisms, and then check the requisite relations. This quickly becomes computationally intensive. We instead proceed via an indirect proof which is more conceptual. The proof strategy will involve computing the dimension of various Hom spaces in $\mathbf{FundRep}(U_q(\mathfrak{sp}_6))$ to argue that there must be a relation between certain webs; we then compose these relations with other webs which allow us to evaluate the webs, giving a necessary condition for the coefficients of the relation. We eventually find enough necessary conditions to solve for the coefficients. A similar proof strategy was used by Kuperberg in [16] to find relations in the $\mathfrak{g}_2$ spider, and in the Ph.D. thesis of Kim [15] to find relations in the $\mathfrak{sl}_4$ spider.

Theorem 3.0.5. There exists an essentially surjective functor $\Psi : \mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{FundRep}(U_q(\mathfrak{sp}_6))$.

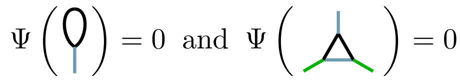
Proof. First, we recall from the background. We have $V \otimes V = \Gamma_{2,0,0} \oplus W \oplus \mathbb{C}(q)$ and $V \otimes W = \Gamma_{1,1,0} \oplus X \oplus V$. Both of these imply that $\mathrm{Hom}_{\mathfrak{sp}_6}(V \otimes V, W)$ and $\mathrm{Hom}_{\mathfrak{sp}_6}(V \otimes W, U)$ are 1-dimensional, by Schur's Lemma. Choosing a non-zero morphism in each of these Hom-spaces, there exists a pivotal functor Ψ from the strict pivotal category freely generated by self-dual objects $\{1, 2, 3\}$ and the morphisms



and


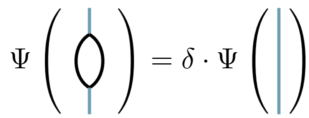
to $\mathbf{FundRep}(U_q(\mathfrak{sp}_6))$ that sends $1 \mapsto V$, $2 \mapsto W$, and $3 \mapsto X$, and sends these webs to the morphisms outlined in Section 2.7. By construction, Ψ is essentially surjective.

It remains to show that Ψ descends to the quotient obtained by imposing the relations in $\mathbf{Web}(\mathfrak{sp}_6)$. First, note that



(3.0.1)

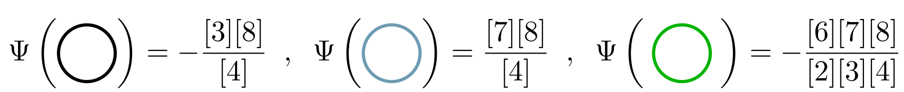
since $\mathrm{Hom}_{\mathfrak{sp}_6}(W, \mathbb{C}(q)) = 0$ and $\mathrm{Hom}_{\mathfrak{sp}_6}(X \otimes X, W) = 0$. Further



for some $\delta \in \mathbb{C}(q)$ since $\mathrm{Hom}_{\mathfrak{sp}_6}(W, W) \cong \mathbb{C}(q)$. We recall the quantum dimensions of V, W, X:

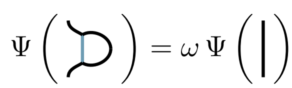
$$\dim_q(V) = \frac{[3][8]}{[4]} \quad , \quad \dim_q(W) = \frac{[7][8]}{[4]} \quad , \quad \dim_q(X) = \frac{[6][7][8]}{[2][3][4]}$$

By our choice of ribbon element (Remark 3.0.4) which acts by $-K_\rho$ on V and X, but K_ρ on W, we have

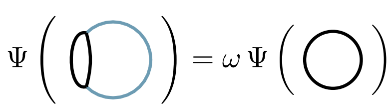


(3.0.2)

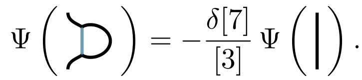
Because $\mathrm{Hom}_{\mathfrak{sp}_6}(W, W)$ is 1-dimensional by Schur's Lemma, we have a relation of the form



"Closing" this relation gives



so that $\delta \frac{[7][8]}{[4]} = \omega \left(-\frac{[3][8]}{[4]} \right)$; thus,

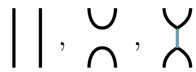


(3.0.3)

Next, the tensor product decomposition

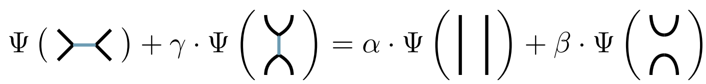
$$V \otimes V = \Gamma_{2,0,0} \oplus W \oplus \mathbb{C}(q)$$

implies that the images under Ψ of

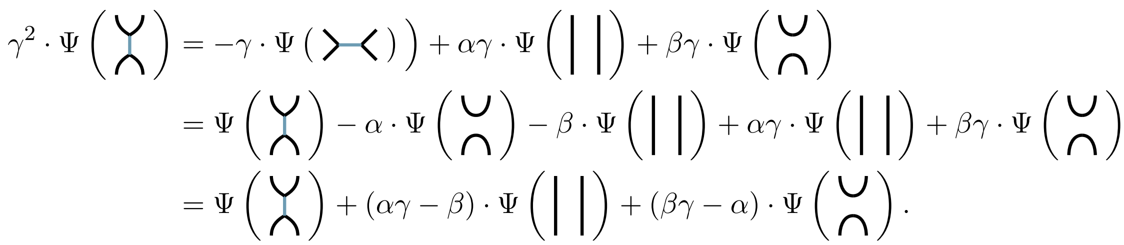


(3.0.4)

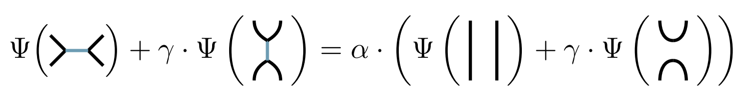
are linearly independent, hence give a basis for $\mathrm{End}_{U_q(\mathfrak{sp}_6)}(V \otimes V)$. Thus, we must have a relation of the form



For some $\alpha, \beta, \gamma \in \mathbb{C}(q)$. Applying this relation twice, we see

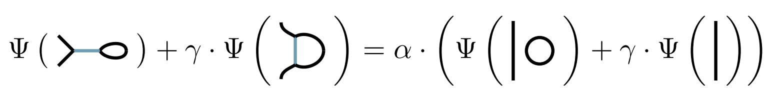


By linear independence, we must have that $\gamma^2 = 1$ and $\beta = \gamma\alpha$. Thus,



(3.0.5)

Closing the 1-labelled edges on the right side, we have



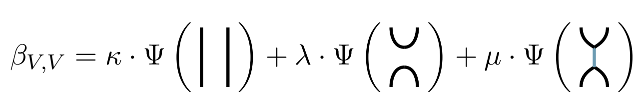
Applying Equations (3.0.1), (3.0.3), (3.0.2), we see that

$$-\gamma\delta\frac{[7]}{[3]} = \alpha\left(-\frac{[3][8]}{[4]} + \gamma\right)$$

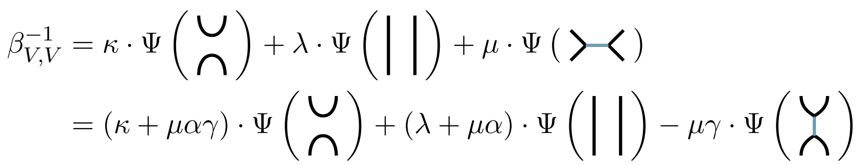
so, recalling that $\frac{[3][8]}{[4]} = [7] - 1$ and $\gamma^2 = 1$,

$$\delta[7] = [3]\alpha\left(\gamma([7]-1) - 1\right). \tag{3.0.6}$$

Next, because $\mathbf{FundRep}(U_q(\mathfrak{sp}_6))$ inherits a braiding β from $\mathbf{Rep}(U_q(\mathfrak{sp}_6))$, it follows that

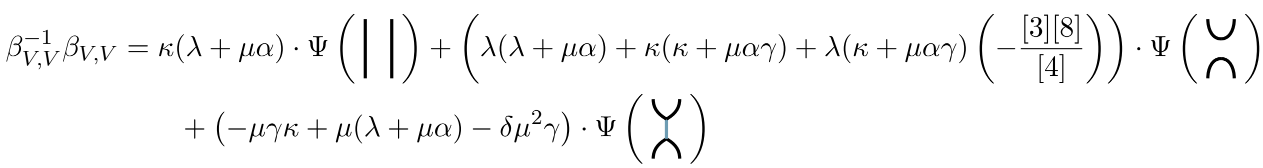


for some $\kappa, \lambda, \mu \in \mathbb{C}(q)$. The self-duality structure then implies that



where in the second line, we have applied Equation (3.0.6).

Expanding $\beta_{V,V}^{-1}\beta_{V,V}$ gives



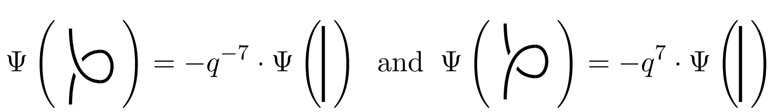
The equality $\beta_{V,V}^{-1}\beta_{V,V} = \mathrm{id}_{V\otimes V}$, together with linear independence of the images of (3.0.4), implies that

$$\begin{aligned} \kappa(\lambda+\mu\alpha) &= 1 \\ \frac{\kappa}{\lambda} + \frac{\lambda+\mu\alpha}{\kappa+\mu\alpha\gamma} &= \frac{[3][8]}{[4]} \\ \lambda+\mu\alpha &= \gamma(\kappa+\mu\delta). \end{aligned} \tag{3.0.7}$$

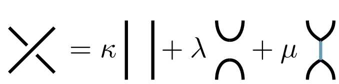
Substituting the left-hand side of the third into the first, we have

$$\kappa\gamma(\kappa+\mu\delta) = 1. \tag{3.0.8}$$

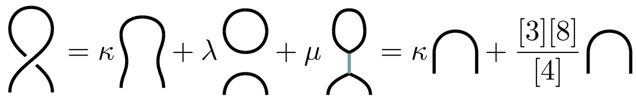
Recall, as in Remark 3.0.4, that we are using the conventions in [32], the ribbon element acts on a representation of highest weight λ via $q^{-(\lambda,\lambda+2\rho)}$, and that we have chosen the ribbon element to act via the *negative* full twist, so the ribbon element acts on V via $-q^{-(\omega_1,\omega_1+2\rho)}$; thus,



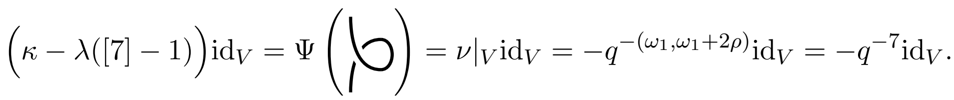
Setting



we have that



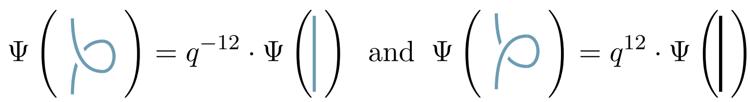
so that



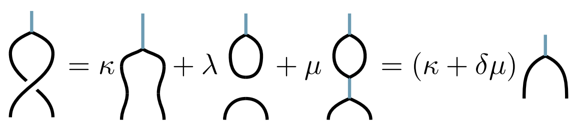
Thus

$$\lambda = \frac{\kappa + q^{-7}}{[7] - 1} \tag{3.0.9}$$

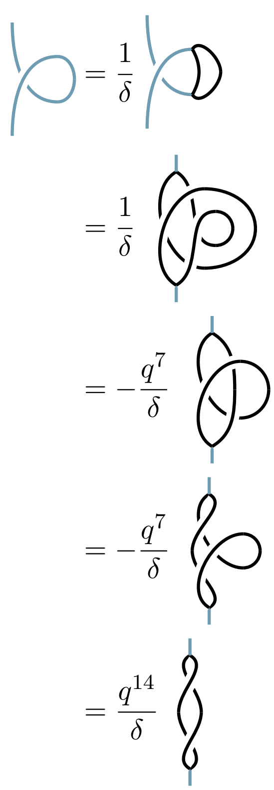
Similarly, the ribbon element acts on W via $q^{-(\omega_1+\omega_2,\omega_1+\omega_2+\rho)} = q^{-12}$, so

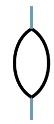


First, note that



So that



$$= \frac{q^{14}(\kappa + \delta\mu)^2}{\delta}$$ 

$$= q^{14}(\kappa + \delta\mu)^2$$

so

$$\kappa + \mu\delta = \pm q^{-1}. \tag{3.0.10}$$

Equations (3.0.8) and (3.0.10) then imply that

$$\kappa = \pm q.$$

We further claim that we have $\kappa \neq -q$. Indeed, if $\kappa = -q$, then (3.0.9) implies that $\lambda|_{q=1} = 0$, $\mu\alpha|_{q=1} = -1$ by (3.0.7). Equation (3.0.7) also gives that $\mu\delta|_{q=1} = 1 - \gamma$. Multiplying (3.0.6) by μ and evaluating at $q = 1$ then gives that

$$7(1 - \gamma) = -3(6\gamma - 1)$$

which implies that $\gamma = -4/11$, contradicting that $\gamma = \pm 1$.

Thus, we have that $\kappa = q$. Equation (3.0.9) then implies that

$$\lambda = q^{-3}\frac{(q^4 + q^{-4})[4]}{[3][8]} = \frac{q^{-3}}{[3]}$$

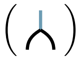
so

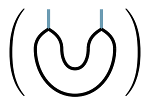
$$\mu\alpha = \kappa^{-1} - \lambda = q^{-1} - \frac{q^{-3}}{[3]} = \frac{[2]}{[3]}.$$

By equation (3.0.7), this implies that $\mu\delta = \gamma q^{-1} - q$, so in particular $\mu\delta|_{q=1} = \gamma - 1$. Multiplying (3.0.6) by μ and evaluating at $q = 1$ as above then gives

$$7(\gamma - 1) = 2(6\gamma - 1)$$

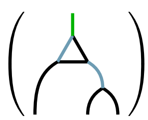
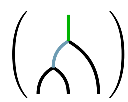
so $\gamma = -1$. This in turn implies that $\mu\delta = -[2]$ and thus $\delta = -[3]\alpha$.

Hence, if we can deduce the value of δ, we will have identified all unknown coefficients, and deduced that all $\mathbf{Web}(\mathfrak{sp}_6)$ that don't involve 3-labeled edges hold (We've also already deduced one relation involving a 3-labeled edge holds.) Note that we don't expect to explicitly identify δ at this point; indeed, there is some flexibility in our choice for this parameter, since changing our choice of Ψ  will change δ by the square of an element in $\mathbb{C}(q)$. We thus can compute δ by choosing $M \in \mathrm{Hom}_{\mathfrak{sp}_6}(V \otimes V, W)$ and explicitly comparing the maps

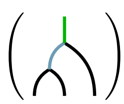
Ψ  and Ψ 

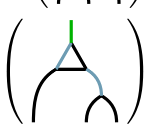
In Section 2.3.3, we defined these maps so that they are equal, and checked that $\delta = -[2][3]$. It then follows that $\alpha = [2]$ and $\mu = \frac{1}{[3]}$.

Next, since $\mathrm{Hom}_{\mathfrak{sp}_6}(V^{\otimes 3}, U)$ is 1-dimensional, we must have

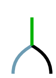
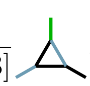
Ψ  $= \tau \cdot \Psi$ 

for some τ. Again, recalling the module maps defined in Section 2.3.3, we may compute both maps on an element $V \otimes V \otimes V$. Explicitly,

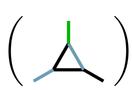
Ψ  $(v_1 \otimes v_2 \otimes v_3) = q[3]^2 x_1$

Ψ  $(v_1 \otimes v_2 \otimes v_3) = -q[2][3]^3 x_1$

so $\tau = -[2][3]$. We thus set

 $:= -\frac{1}{[2][3]}$  .

Indeed, recalling the maps defined in Section 2.3.3, we have

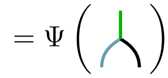
$-\frac{1}{[2][3]}\Psi$  $(w_{12} \otimes v_3)$

$$= -\frac{1}{[2][3]}(q_X \circ (id_V \otimes p_W))(qv_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_1 \otimes v_3)$$

$$= -\frac{1}{[2][3]} q_X(-q[3]v_1 \otimes w_{23} + [3]v_2 \otimes w_{13})$$

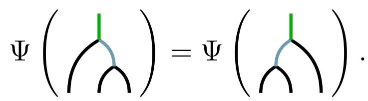
$$= -\frac{1}{[2][3]}(q^2[3]^2 x_1 + [3]^2 x_1)$$

$$= -q[3]x_1$$



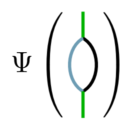
$$(w_{12} \otimes v_3).$$

which implies that



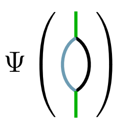
(3.0.11)

Since $\mathrm{End}_{\mathfrak{sp}_6}(U)$ is 1-dimensional, we have



$$= \delta' \cdot \mathrm{id}_U.$$

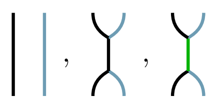
Recalling our maps defined in Chapter 2, we have



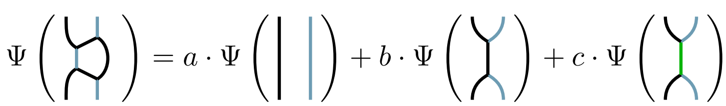
$$(x_1) = p_X(q^{-1}w_{23} \otimes v_1) - w_{13} \otimes v_2 + q\, w_{12} \otimes v_3) = -q^{-2}x_1 - [3]x_1 - q^2 x_1 = -[3]^2 x_1$$

so that $\delta' = -[3]^2$.

For our final two relations, we begin by noting that the images of



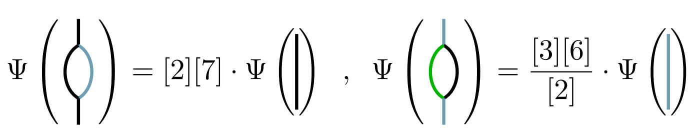
give a basis for $\mathrm{End}_{\mathfrak{sp}_6}(V \otimes W)$. This implies that



(3.0.12)

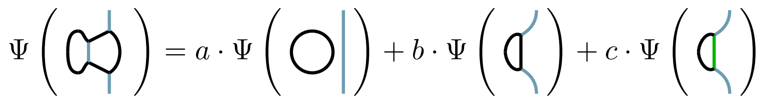
for some $a, b, c \in \mathbb{C}(q)$.

First, applying the same trick used to find (3.0.3) and knowing that $\delta = -[2][3], \delta' = -[3]^2$, we see that



(3.0.13)

Closing the leftmost strands of Equation (3.0.12), we have

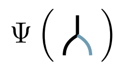


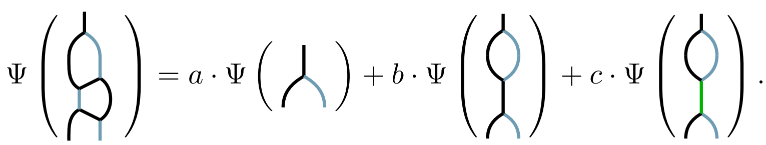
giving

$$[2][7](-[2][3]) = -\frac{[3][8]}{[4]}a - [2][3]b + \frac{[3][6]}{[2]}c$$

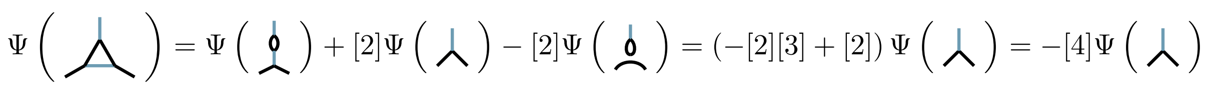
so that

$$[2]^2[7] = \frac{[8]}{[4]}a + [2]b - \frac{[6]}{[2]}c.$$

Now, composing (3.0.12) with  gives

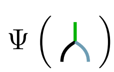


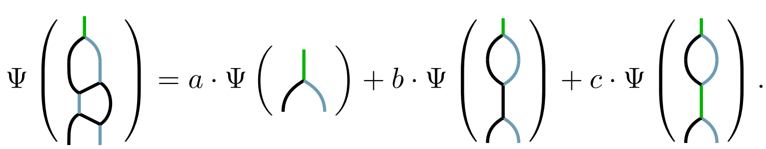
Next, we use the fact that



we obtain the equation

$$[4]^2 = a + [2][7]b, .$$

Similarly, composing (3.0.12) with  gives



giving

$$[2]^2[3]^2 = a - [3]^2c.$$

Solving the system of equations

$$[2]^2[7] = \frac{[8]}{[4]}a + [2]b - \frac{[6]}{[2]}c$$

$$[4]^2 = a + [2][7]b$$

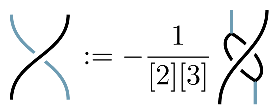
$$[2]^2[3]^2 = a - [3]^2 c$$

gives that

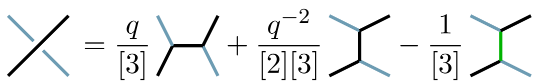
$$a = [3]^2 \ , \ b = \frac{1}{[2]} \ , \ c = -[3]$$

as desired.

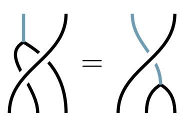
Finding the last relation will be done slightly differently, although similar methods are possible. In the next section[1], we shall compute that if we define



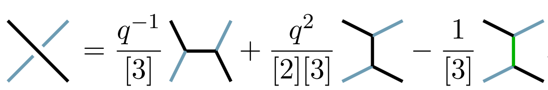
then we have



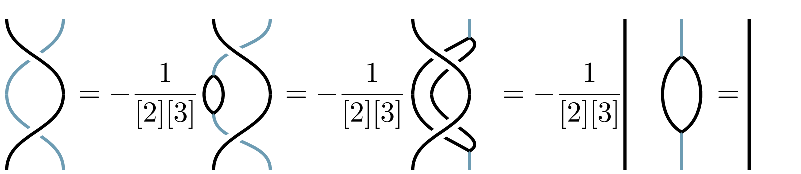
which satisfies the "fork slide relation"



One can check that this has inverse

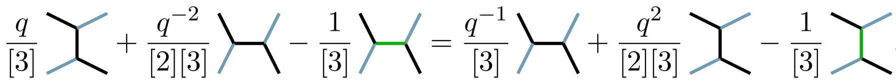


But we know (from facts about 1 colored crossings) that the top crossing has inverse given by rotating the given formula by 90 degrees, because

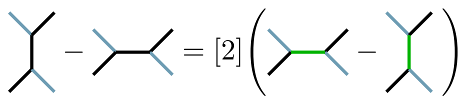


[1]We do **not** use this identity when computing this or leading up to it

Thus we have the equality



Simplifying the above equation gives



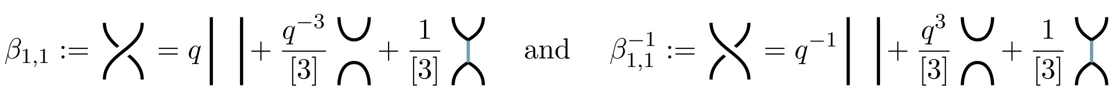
∎

In fact, we make the conjecture

Conjecture 3.0.6. The functor $\Psi : \mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{FundRep}(U_q(\mathfrak{sp}_6))$ is an equivalence of categories.

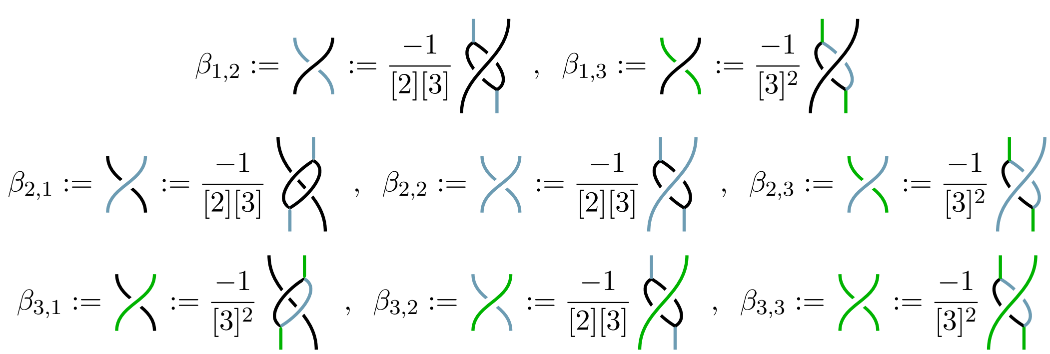
3.1 The braiding on $\mathbf{Web}(\mathfrak{sp}_6)$

The proof of Theorem 3.0.5 suggests that we can define a braided structure on $\mathbf{Web}(\mathfrak{sp}_6)$ by setting



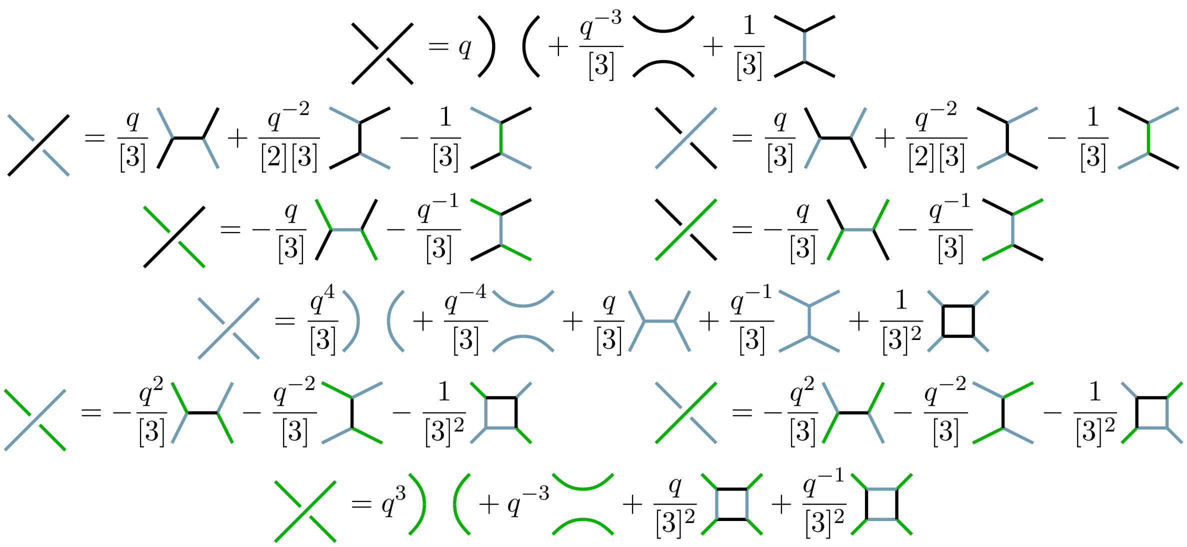
(3.1.1)

The proof of Theorem 3.0.5 shows that we indeed have $\beta_{1,1}\beta_{1,1}^{-1} = ||$. Naturality of the braiding implies that we must set



(3.1.2)

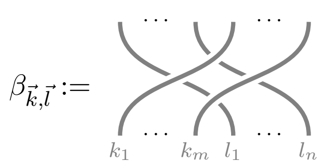
These assignments then determine $\beta_{k,l}^{-1}$ for $k, l \in \{1, 2, 3\}$ using the pivotal structure. Computing these, we find the following explicit formulas:



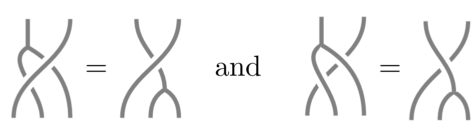
(3.1.3)

Theorem 3.1.1. The formulae in equation (3.1.3) endow $\mathbf{Web}(\mathfrak{sp}_6)$ with the structure of a ribbon category.

Proof. To begin, we extend the definition of β to all objects $\vec{k} = (k_1, \ldots, k_m)$ and $\vec{l} = (l_1, \ldots, l_n)$ in $\mathbf{Web}(\mathfrak{sp}_6)$ by setting

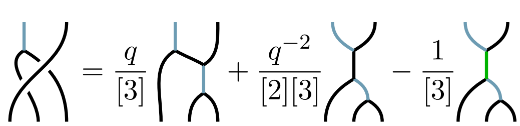


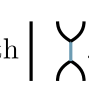
which also determines β^{-1} using the pivotal structure. First, one can check check that the "fork slide" relation holds:



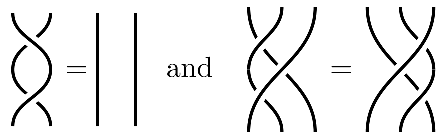
(3.1.4)

for all valid colorings of the edges. For example, in the proof of Theorem 3.0.5, we saw that



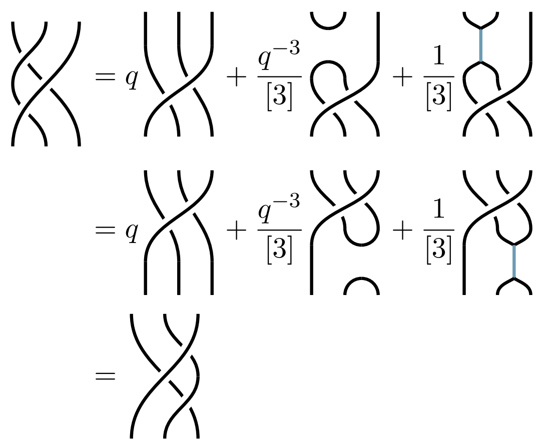
which gives the first relation in (3.1.4) in this case by composing with .

The braid relations then follow from the 1-labeled case, i.e. from the relations

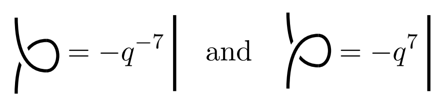


(3.1.5)

The first (Reidemeister II) of these relations holds since the coefficients in (3.1.1) satisfy (3.0.7), while the second (Reidemeister III) is given as follows:



Finally, we also have that



(3.1.6)

so $\mathbf{Web}(\mathfrak{sp}_6)$ is ribbon (since $(-q^{-7})^{-1} = -q^{7}$). □

Further, it's clear from the proof of Theorem 3.0.5 that the functor Ψ defined therein is ribbon.

CHAPTER 4

$\mathbf{Web}(\mathfrak{sp}_{2n})$ and the BMW Algebra

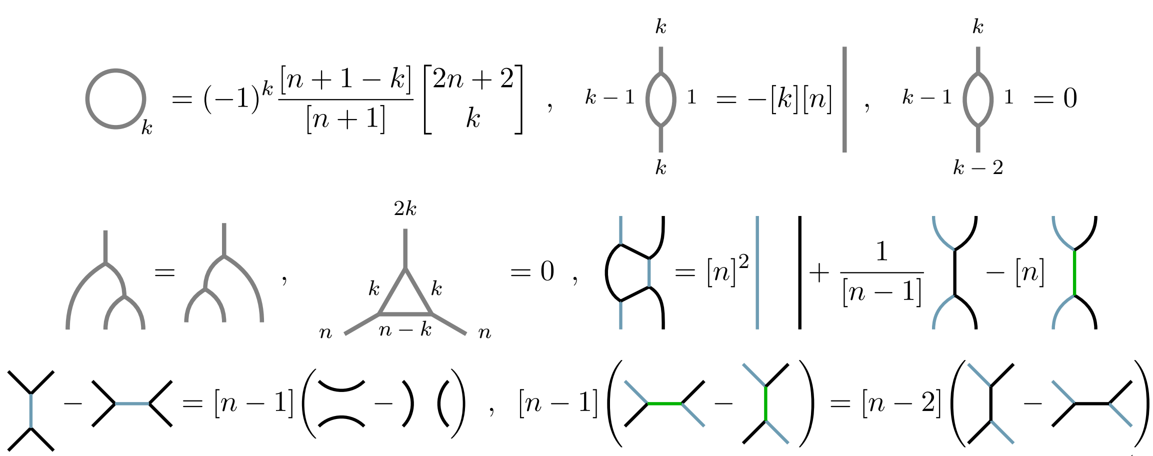
To aid in showing that Ψ is full, we first establish the relation between $\mathbf{Web}(\mathfrak{sp}_6)$ and the BMW algebra [3],[21].

In fact, our results in this section actually extend to a relation between the BMW algebra and a family of categories which we denote $\mathbf{Web}(\mathfrak{sp}_{2n})$. Although $\mathbf{Web}(\mathfrak{sp}_{2n})$ agrees with $\mathbf{Web}(\mathfrak{sp}_6)$ when $n = 3$, we do not posit an extension of Conjecture 3.0.6; we believe there are more relations needed in $\mathbf{Web}(\mathfrak{sp}_{2n})$.

Definition 4.0.1. Let $\mathbf{Web}(\mathfrak{sp}_{2n})$ be the strictly pivotal $\mathbb{C}(q)$-linear category generated by self-dual objects $\{1, 2, 3, \ldots, n\}$ and with morphisms generated by

$$\in \mathrm{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(k \otimes \ell, k + \ell)$$

modulo the local relations:



(4.0.1)

Note that at $n = 2$ and $n = 3$, we recover the $\mathbf{Web}(\mathfrak{sp}_{2n})$ categories we already know (and in fact, at $n = 1$, we recover a specialization of the Temperly-Lieb Category, $\mathbf{Web}(\mathfrak{sp}_2)$).

First, we recall the definition of the BMW algebra, following the conventions from [10].

Definition 4.0.2. The Birman-Murakami-Wenzl algebra, i.e. the BMW algebra $\mathrm{BMW}_k(r, z)$ is the unital, associative $\Bbbk$-algebra generated by e_i, g_i, g_i^{-1} for $1 \le i \le k-1$, with relations:

1. $g_i - g_i^{-1} = z(1 - e_i)$
2. $e_i^2 = \left(1 + \frac{r - r^{-1}}{z}\right) e_i$
3. $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $1 \le i \le k-2$
4. $g_i g_j = g_j g_i$ for $|i - j| > 1$
5. $e_i e_{i+1} e_i = e_i$ and $e_{i+1} e_i e_{i+1} = e_{i+1}$ for $1 \le i \le k-2$
6. $g_i g_{i+1} e_i = e_{i+1} e_i$ and $g_{i+1} g_i e_{i+1} = e_i e_{i+1}$ for $1 \le i \le k-2$
7. $e_i g_i = g_i e_i = r^{-1} e_i$
8. $e_i g_{i+1} e_i = r e_i$ and $e_{i+1} g_i e_{i+1} = r e_{i+1}$ for $1 \le i \le k-2$.

We note that there is some redundancy in the relations, but we will stick with the above list.

At the specializations $r = -q^{2n+1}, z = q - q^{-1}$, the BMW algebra is "quantum Schur-Weyl" dual to $U_q(\mathfrak{sp}_{2n})$, see [10]. In particular, there is a surjective homomorphism

$$\mathrm{BMW}_k(-q^n, q - q^{-1}) \to \mathrm{End}_{U_q(\mathfrak{sp}_{2n})}(V^{\otimes k}). \tag{4.0.2}$$

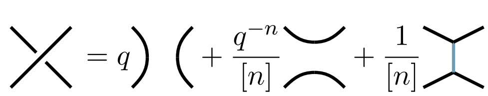
We prove the following partial analogue

Theorem 4.0.3. There is a homomorphism

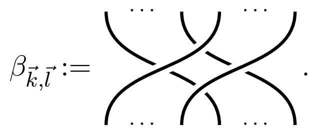
$$\mathrm{BMW}(-q^{2n+1}, q - q^{-1}) \to \mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_{2n})}(1^{\otimes k}).$$

Recall that specializing $\mathbf{Web}(\mathfrak{sp}_{2n})$ at $n = 2, 3$ recovers $\mathbf{Web}(\mathfrak{sp}_4)$, $\mathbf{Web}(\mathfrak{sp}_6)$, respectively.

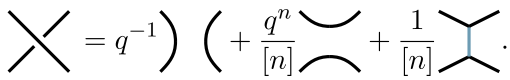
Lemma 4.0.4. The full subcategory of $\mathbf{Web}(\mathfrak{sp}_{2n})$ generated by 1 is braided, with braiding defined by



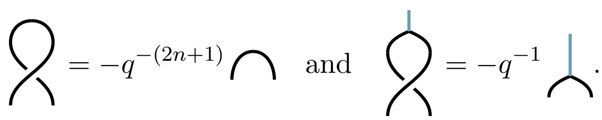
extended to all objects by setting



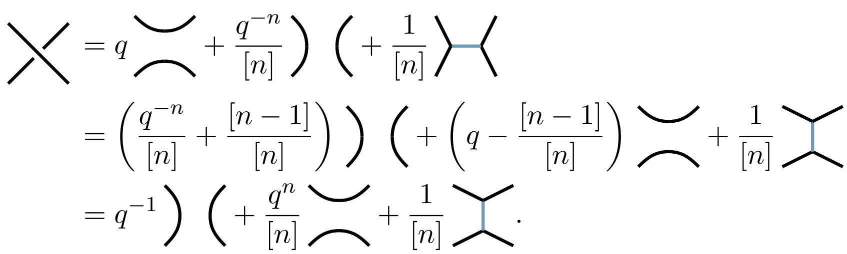
This braiding has inverse



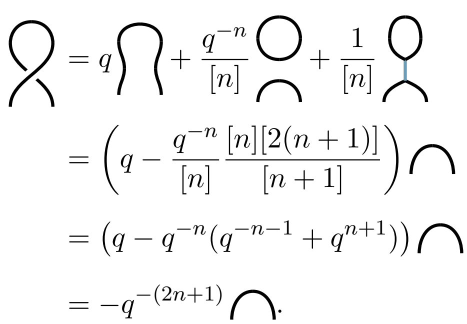
Further, the braiding satisfies



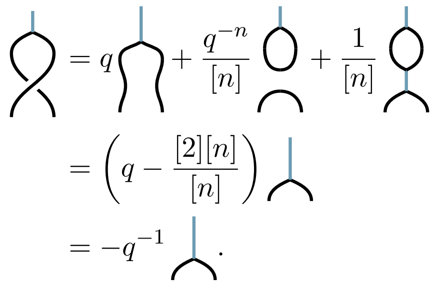
Proof. First, we compute that



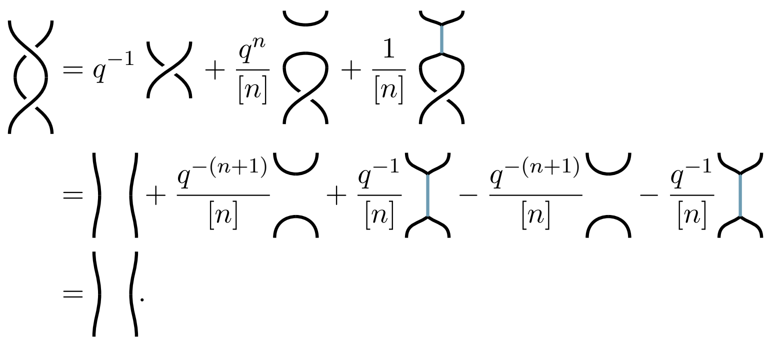
Now we record that



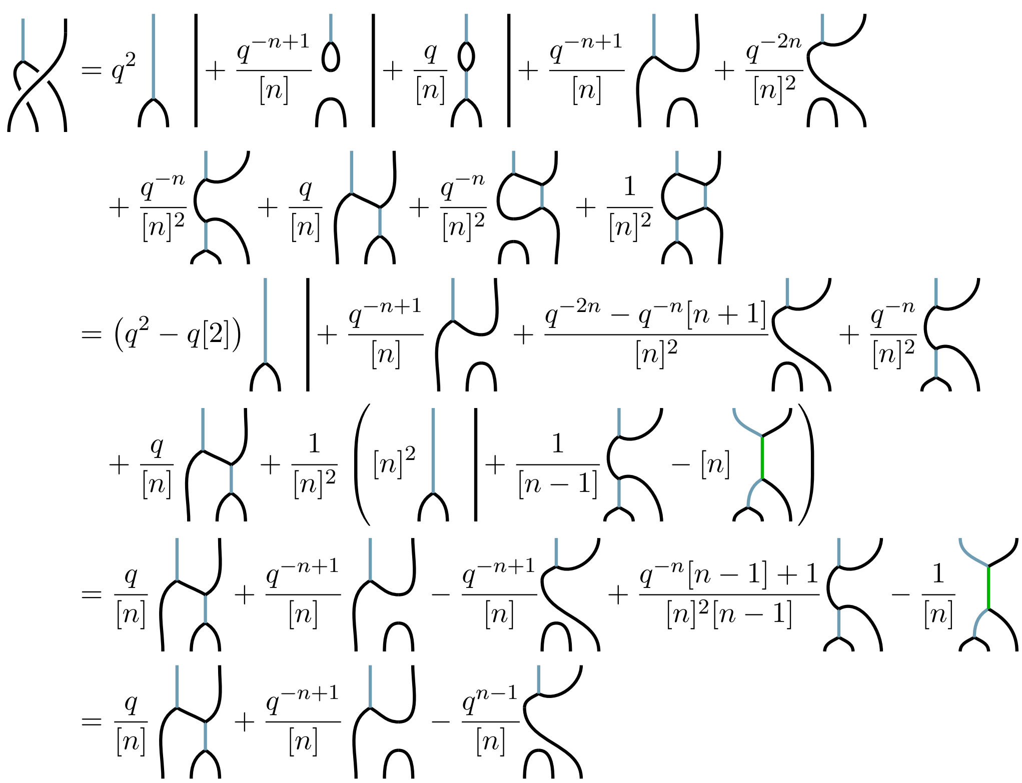
It will also be useful to record that

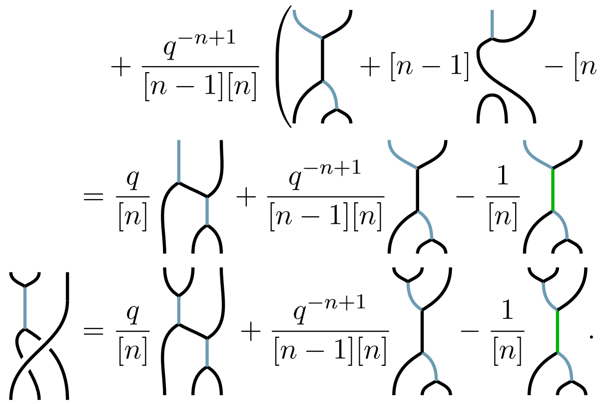


We now see that

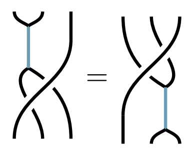


To see that the braid relation holds, we first compute

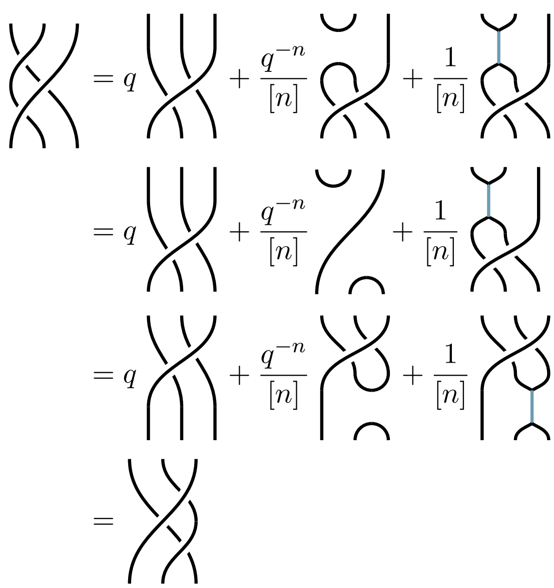




By rotational symmetry, we have



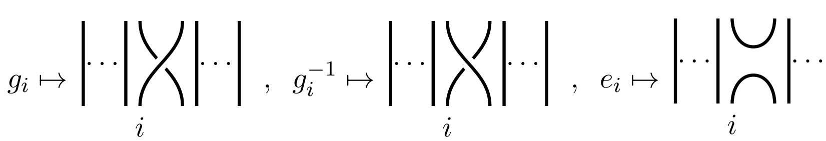
and thus compute



so that the braid relation holds. ∎

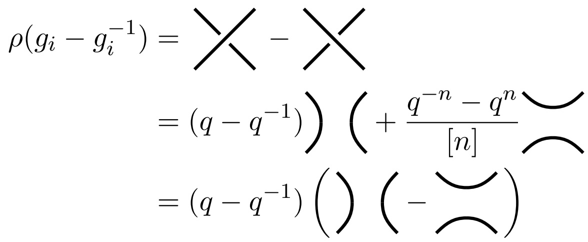
We now prove Theorem 4.0.3.

Proof. Let $r = -q^{2n+1}$ and $z = q - q^{-1}$, define the map $\rho : \mathrm{BMW}(r, z) \to \mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_{2n})}(1^{\otimes k})$ by



where the indices above are **not** labels of the strands, but rather denote the i^{th} term of 1 in $1^{\otimes k}$. We show this map factors through the relations of the BMW algebra, as enumerated in Definition 4.0.2.

1. $g_i - g_i^{-1} = z(1 - e_i)$. We compute

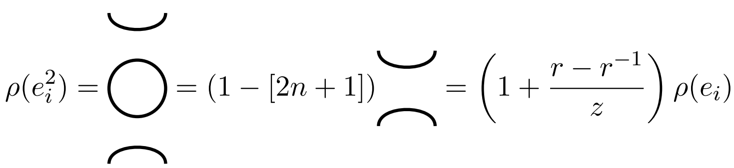


$$= z\rho(1 - e_i)$$

2. $e_i^2 = \left(1 + \frac{r - r^{-1}}{z}\right) e_i$. Note that

$$1 + \frac{r - r^{-1}}{z} = 1 + \frac{-q^{2n+1} + q^{-(2n+1)}}{q - q^{-1}} = 1 - [2n+1]$$

so that

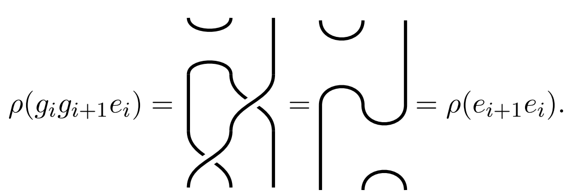


3. $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, this holds by Reidemeister III invariance of the crossing in $\mathbf{Web}(\mathfrak{sp}_{2n})$.

4. $g_i g_j = g_j g_i$ for $|i - j| > 1$, this holds by isotopy in $\mathbf{Web}(\mathfrak{sp}_{2n})$.

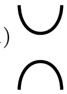
5. $e_i e_{i+1} e_i = e_i$ and $e_{i+1} e_i e_{i+1} = e_{i+1}$, this again holds by isotopy in $\mathbf{Web}(\mathfrak{sp}_{2n})$.

6. $g_i g_{i+1} e_i = e_{i+1} e_i$ and $g_{i+1} g_i e_{i+1} = e_i e_{i+1}$. We have



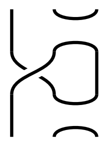
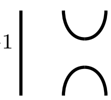
The second equation is similar.

7. $e_ig_i = g_ie_i = r^{-1}e_i$. We have

$\rho(e_ig_i) =$  $= -q^{-(2n+1)}$  $= r^{-1}\rho(e_i)$

and similarly $\rho(g_ie_i) = -q^{2n+1}\rho(e_i)$.

8. $e_ig_{i+1}e_i = re_i$ and $e_{i+1}g_ie_{i+1} = re_{i+1}$. The first is given by

$\rho(e_ig_{i+1}e_i) =$  $= -q^{2n+1}$  $= r\rho(e_i)$

and the second is similar.

□

CHAPTER 5

Fullness of $\Psi : \mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{FundRep}(U_q(\mathfrak{sp}_6))$

We now prove

Theorem 5.0.1. The functor $\Psi : \mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{FundRep}(U_q(\mathfrak{sp}_6))$ is full.

Proof. The strategy is to first show that

$$\Psi : \mathrm{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(1^{\otimes r}, 1^{\otimes s}) \to \mathrm{Hom}_{U_q(\mathfrak{sp}_6)}(V^{\otimes r}, V^{\otimes s}) \tag{5.0.1}$$

is surjective for all $r, s \geq 0$. We will then use the fact that W and X are irreducible summands of $V \otimes V$ and $V \otimes V \otimes V$ (respectively) to extend (5.0.1) to the tensor product of any finite number of irreducible representations.

First, the homomorphism $\mathrm{BMW}_k(-q^7, q - q^{-1}) \to \mathrm{End}_{\mathfrak{sp}_6}(V^{\otimes k})$ is known to be surjective [10]. Because the diagram

$$\begin{array}{ccc} \mathrm{BMW}_k(-q^7, q - q^{-1}) & \longrightarrow & \mathrm{End}_{U_q(\mathfrak{sp}_6)}(V^{\otimes k}) \\ & \searrow \quad \nearrow_{\Psi} & \\ & \mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_6)}(1^{\otimes k}) & \end{array}$$

commutes and the top arrow is surjective, Ψ is surjective as well. Now, suppose $r + s$ (where r, s are as in (5.0.1)) is odd. Then we have that both $\mathrm{Hom}_{U_q(\mathfrak{sp}_6)}(V^{\otimes r}, V^{\otimes s})$ and $\mathrm{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(1^{\otimes r}, 1^{\otimes s})$ are trivial (the former is a fact about $U_q(\mathfrak{sp}_6)$, the latter holds because all morphisms in $\mathbf{Web}(\mathfrak{sp}_6)$ preserve parity of the sums of the labels).

Now suppose $r + s$ is even. The cap and cup webs in $\mathbf{Web}(\mathfrak{sp}_6)$ and the unit/counit morphisms in**FundRep**$(U_q(\mathfrak{sp}_6))$ give the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(1^{\otimes r}, 1^{\otimes s}) & \xrightarrow{\Psi} & \mathrm{Hom}_{\mathfrak{sp}_6}(V^{\otimes r}, V^{\otimes s}) \\ \big\downarrow \cong & & \big\downarrow \cong \\ \mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_6)}(1^{\otimes \frac{r+s}{2}}) & \xrightarrow{\Psi} & \mathrm{End}_{\mathfrak{sp}_6}(V^{\otimes \frac{r+s}{2}}) \end{array}$$

Thus,

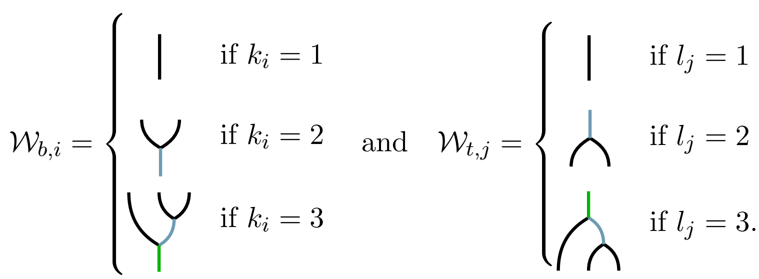
$$\Psi : \mathrm{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(1^{\otimes r}, 1^{\otimes s}) \to \mathrm{Hom}_{\mathfrak{sp}_6}(V^{\otimes r}, V^{\otimes s})$$

is surjective for all $r, s \geq 0$.

It remains to extend this to the remaining Hom-spaces in $\mathbf{Web}(\mathfrak{sp}_6)$. Let $\vec{k} = k_1 \otimes \cdots \otimes k_m$ and $\vec{l} = l_1 \otimes \cdots \otimes l_n$ be objects in $\mathbf{Web}(\mathfrak{sp}_6)$. Consider the webs

$$\mathcal{W}_b = \bigotimes_{i=1}^{m} \mathcal{W}_{b,i} : (k_1, \ldots, k_m) \to 1^{\otimes \sum k_i} \quad \text{and} \quad \mathcal{W}_t = \bigotimes_{j=1}^{n} \mathcal{W}_{t,j} : 1^{\otimes \sum l_j} \to (l_1, \ldots, l_n)$$

defined by

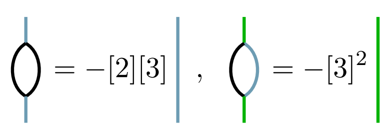


So $\mathcal{W}_{b,i}$ and $\mathcal{W}_{t,j}$ include and project each k_i, l_j into/onto a tensor product of 1.

We then obtain a $\mathbb{C}(q)$-linear map

$$\mathrm{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(1^{\otimes \sum k_i}, 1^{\otimes \sum l_j}) \xrightarrow{\mathcal{W}_t \circ (-) \circ \mathcal{W}_b} \mathrm{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(\vec{k}, \vec{l})$$

which is surjective, because we may apply the relations



"in reverse" (which we call "exploding digons") to write a web in the image of $\mathcal{W}_t \circ (-) \circ \mathcal{W}_b$. This in turn implies that

$$\mathrm{Hom}_{\mathfrak{sp}_6}(V^{\otimes \sum k_i}, V^{\otimes \sum l_j}) \xrightarrow{\Psi(\mathcal{W}_t)\circ(-)\circ\Psi(\mathcal{W}_b)} \mathrm{Hom}_{\mathfrak{sp}_6}(\Psi(\vec{k}), \Psi(\vec{l}))$$

is surjective as well, and the result then follows from the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(1^{\otimes \sum k_i}, 1^{\otimes \sum l_j}) & \xrightarrow{\Psi} & \mathrm{Hom}_{\mathfrak{sp}_6}(V^{\otimes \sum k_i}, V^{\otimes \sum l_j}) \\ \downarrow{\scriptstyle \mathcal{W}_t\circ(-)\circ\mathcal{W}_b} & & \downarrow{\scriptstyle \Psi(\mathcal{W}_t)\circ(-)\circ\Psi(\mathcal{W}_b)} \\ \mathrm{Hom}_{\mathbf{Web}(\mathfrak{sp}_6)}(\vec{k}, \vec{l}) & \xrightarrow{\Psi} & \mathrm{Hom}_{\mathfrak{sp}_6}(\Psi(\vec{k}), \Psi(\vec{l})) \end{array}$$

□

CHAPTER 6

$\mathbf{Lad}(\mathfrak{sp}_6)$

In the remainder of the paper, we use a new category $\mathbf{Lad}(\mathfrak{sp}_6)$, which is similar in flavor to $\mathbf{Web}(\mathfrak{sp}_6)$, but with more "rigidity," which will be useful in our arguments. Our motivation comes from the type A analogue. In [4], where it was proved that $\mathbf{Web}(\mathfrak{sl}_n) \cong \mathbf{FundRep}(U_q(\mathfrak{sl}_n))$ a key ingredient was introduction of *ladders* a more "rigid" version of webs, along with *skew Howe duality.* We do not go into detail about skew Howe duality here, but do remark that it has been shown to *not* work in types BCD in [30], at least not in a straightforward way.

Nevertheless, we still take inspiration from the type A story, and introduce a type C analogue of $\mathfrak{gl}_n$ webs, called $\mathbf{Web}(\mathfrak{gsp}_6)$. Paralleling the type A case, the category $\mathbf{Web}(\mathfrak{gsp}_6)$ is related to the category of representations of (quantum) $\mathfrak{gsp}_6$, the Lie algebra of so-called "symplectic similitude group" (c.f. [36])

$$GSp(6) = \{A \in M_6(\mathbb{C}) \mid A^T J A = \lambda_A J\}$$

where J is non-degenerate and skew symmetric and $\lambda_A \in \mathbb{C}$.

Remark 6.0.1. As Lie algebras (over $\mathbb{C}$), $\mathfrak{gsp}_{2n} \cong \mathfrak{sp}_{2n} \times \mathbb{C}$.

Proof. The proof is completely analogous to the type A case, $\mathfrak{gl}_n = \mathfrak{sl}_n \times \mathbb{C}$, as found in [9]. Define $\phi : \mathfrak{gsp}_{2n} \to \mathbb{C}$ by $\phi(X) = \mathrm{tr}(X)$. By properties of the trace, ϕ is linear and $\phi([X,Y]) = 0$, so ϕ is a map of Lie algebras. Then $\ker(\phi) = \mathfrak{sp}_{2n}$ so we have a short exact sequence

$$0 \to \mathfrak{sp}_{2n} \to \mathfrak{gsp}_{2n} \to \mathbb{C} \to 0.$$

In fact, there is a splitting $s : \mathbb{C} \to \mathfrak{gsp}_{2n}$ defined by

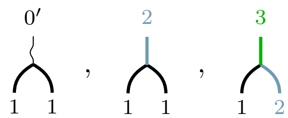
$$s(z) = \frac{z}{2n} I_{2n}$$

where I_n is the $n \times n$ identity matrix. The splitting s has image all scalar matrices, which is disjoint from $\mathfrak{sp}_{2n}$, so

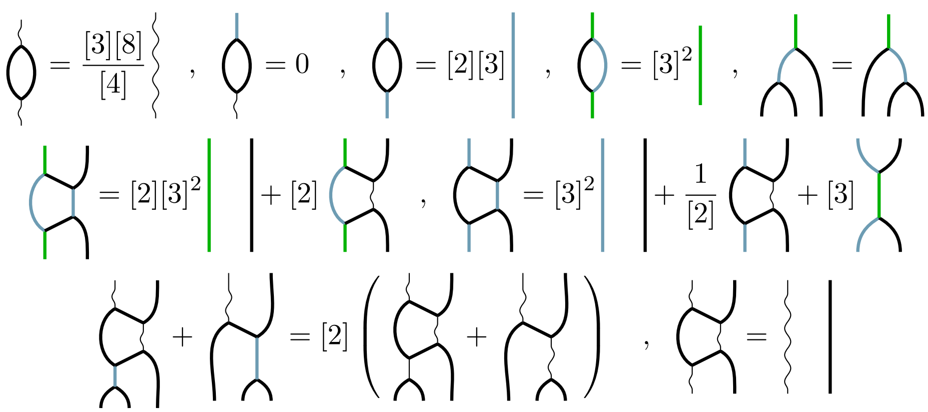
$$\mathfrak{gsp}_{2n} \cong \mathfrak{sp}_{2n} \times \mathbb{C}.$$

$\square$

Definition 6.0.2. The category **Web** $(\mathfrak{gsp}_6)$ is the monoidal category generated by objects $\{0', 1, 2, 3\}$ and morphisms



along with those obtained from these via vertical and horizontal reflection. These generators are subject to the following relations:



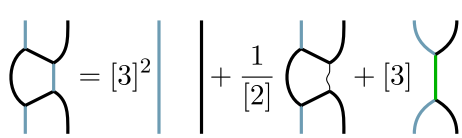
(6.0.1)

again, together with those obtained from these via horizontal and vertical reflection.

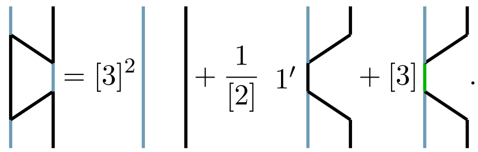
Remark 6.0.3. Note that, by definition, **Web**$(\mathfrak{gsp}_6)$ is not a pivotal category.

Remark 6.0.4. The above relations look similar to those of **Web**$(\mathfrak{sp}_6)$, but with different signs. Indeed, in $\mathfrak{gsp}_6$, we have re-scaled all the "merge" maps by -1, in an attempt to write all positive coefficients of the relations, anticipating an attempt at categorification.

We are almost ready to define the ladder category, but let us pause to highlight why. As we shall see, we would like a category in which all "squares" in the relations above can be written in terms of webs with fewer vertices or with the slanted edges in reverse order. For example, the relation



almost satisfies this, but the second term on the right still has a square. To this end, we will define a new object $1' \cong 1 \otimes 0'$ and amalgamate the two edges, so the relation is instead

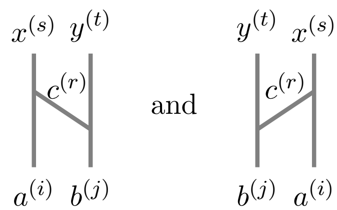


We are now prepared to make the definition of the ladder category. Note that we undo the rescaling of vertices which we made to put $\mathbf{Web}(\mathfrak{gsp}_6)$ in a more categorifiable form.

Definition 6.0.5. The category $\mathbf{Lad}(\mathfrak{sp}_6)$ is the $\mathbb{C}(q)$-linear monoidal category with objects generated by:

$$\{0^{(i)}, 1^{(j)}, 2^{(s)}, 3^{(t)} \mid i, j, s, t \geq 0\}.$$

Defining the *mass* of an object by $\mu(k^{(i)}) = k + 2i$, the morphisms are then generated by the *rung morphisms*:



(6.0.2)

where the labels satisfy all of the following conditions:

1. **Rung mass:** $\mu(c^{(r)}) > 0$;
2. **Upper vertex:** exactly one of the following holds for the elements a, c, x:
    - two are equal and the other is 0,
    - two are 1 and the other is 2, or
    - all are nonzero and distinct;
3. **Lower vertex:** exactly one of the following holds for the elements b, c, y:
    - two are equal and the other is 0,
    - two are 1 and the other is 2, or
    - all are nonzero and distinct;

4. **Mass preservation:** $\mu\left(a^{(i)}\right)+\mu\left(c^{(r)}\right)=\mu\left(x^{(s)}\right)$ and $\mu\left(b^{(j)}\right)=\mu\left(y^{(t)}\right)+\mu\left(c^{(r)}\right)$.

Remark 6.0.6. We note that, using the same notation as in Equation 6.0.2,

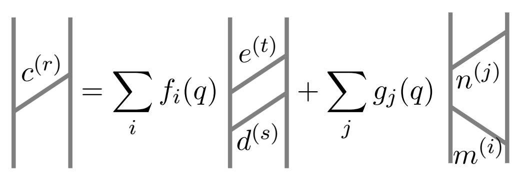
$$\mu\left(a^{(i)}\right)+\mu\left(b^{(j)}\right)=\mu\left(a^{(i)}\right)+\mu\left(c^{(r)}\right)+\mu\left(y^{(s)}\right)=\mu\left(x^{(s)}\right)+\mu\left(y^{(t)}\right)$$

so the total mass of a ladder is the same between rungs.

We adopt terminology from [4]. Specifically, we will refer to compositions of tensor products of generating morphisms as *ladders*, so morphisms in $\mathbf{Lad}(\mathfrak{sp}_6)$ are $\mathbb{C}(q)$-linear combinations of ladders. The vertical line segments in ladders are called *uprights*, and the segments passing between the uprights as *rungs*. We refer to the generating ladders in (6.0.2) as E*-rungs* and F*-rungs*, respectively. Finally, we will write the object (or edge label) $\ell^{(k)}$ as ℓ with k primes, when k is small, e.g. $2'=2^{(1)}$ and $3=3^{(0)}$.

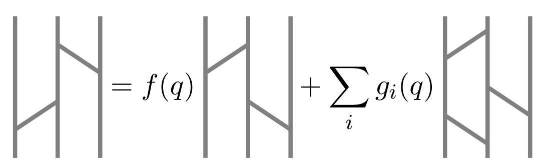
The morphisms are subject to relations all of which take the following forms, or a reflection thereof. Here, we allow some rungs to have zero mass, with the understanding that such a rung is simply the corresponding identity morphism:

1. **Rung explosion:**



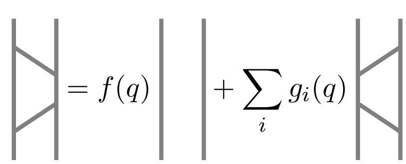
for $f\in\mathbb{C}(q)$ and $\mu\left(c^{(r)}\right)>\max\left(\mu\left(d^{(s)}\right),\mu\left(e^{(t)}\right)\right)$ and $\mu\left(c^{(r)}\right)>\max\left(\mu\left(m^{(i)}\right),\mu\left(n^{(j)}\right)\right)$.

2. **Rung swap:**



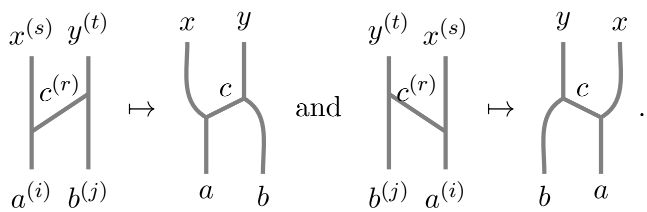
where $f, g_i\in\mathbb{C}(q)$

3. **Square relation:**



The precise relations depend on the labels and masses of the rungs and uprights; are recorded in Appendix A.1. All are “ladderized” versions of $\mathbf{Web}(\mathfrak{sp}_6)$ relations.

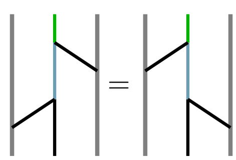
Definition 6.0.7. There is a monoidal functor $\Phi : \mathbf{Lad}(\mathfrak{sp}_6) \to \mathbf{Web}(\mathfrak{sp}_6)$ that, on objects, sends $x^{(i)}$ to x (where we ignore any 0 label). On morphisms, Φ forgets about the rigidity of the ladder structure; in other words, Φ sends



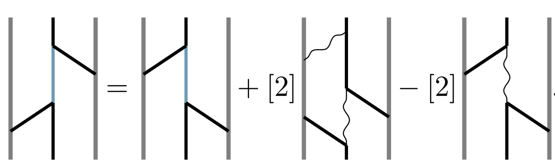
Remark 6.0.8. We content ourselves here with only displaying the general form of the relations in $\mathbf{Lad}(\mathfrak{sp}_6)$, as the specific form of the relation is only used to show that the functor $\Phi : \mathbf{Lad}(\mathfrak{sp}_6) \to \mathbf{Web}(\mathfrak{sp}_6)$ below is well-defined. After that, the general forms of the relations above suffice for the proofs.

The vertices of a laddders are modeled on those of $\mathfrak{sp}_6$ webs, but we have some additional structure. For example, the mass of an edge “keeps track of a tensor product of 1s giving rise to it.”

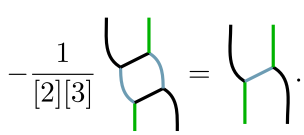
One can check that each of the relations in $\mathbf{Lad}(\mathfrak{sp}_6)$ maps under Φ to a relation in $\mathbf{Web}(\mathfrak{sp}_6)$. There are many to check, but we do an example of each type here. Rung swaps correspond to associativity relations



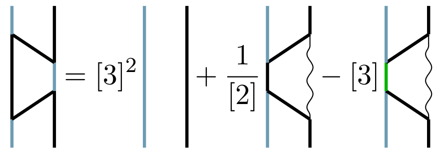
or relations of the form



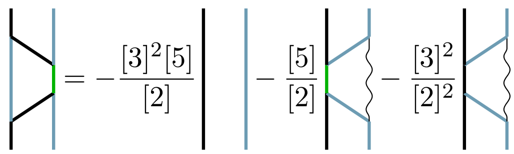
Rung explosions follow by associativity and evaluating digons, such as



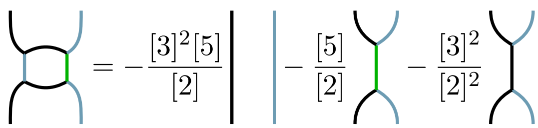
Finally, square relations have more variety. Some correspond exactly to relations in $\mathbf{Web}(\mathfrak{sp}_6)$, such as



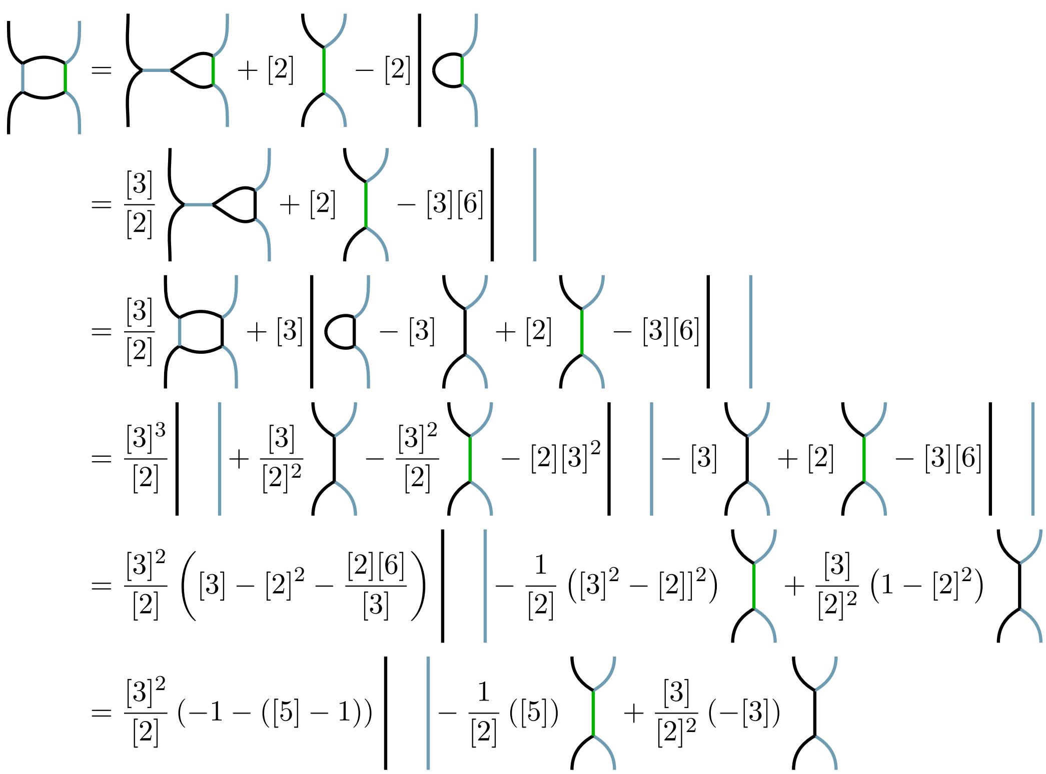
while others are more involved computations, such as the $\mathbf{Lad}(\mathfrak{sp}_6)$ relation

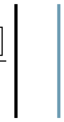
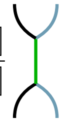
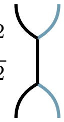


corresponding to the identity



which can be verified via $\mathbf{Web}(\mathfrak{sp}_6)$ relations:



$= -\frac{[3]^2[5]}{[2]}$  $- \frac{[5]}{[2]}$  $- \frac{[3]^2}{[2]^2}$ 

The remaining arguments in this work proceed by "pulling back a web to a ladder," so we begin with a few useful facts about this process.

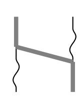
Lemma 6.0.9. Let λ_1, λ_2 be ladders (not necessarally with the same number of uprights) and $\Phi : \mathbf{Lad}(\mathfrak{sp}_6) \to \mathbf{Web}(\mathfrak{sp}_6)$ be the functor defined in Definition 6.0.7 (which forgets the ladder structure, primes, and 0 edeges). Suppose that $\Phi(\lambda_2) \circ \Phi(\lambda_1)$ is defined (in other words, after forgetting all 0 edges and primes, the domain of λ_2 equals the codomain of λ_1). Then there exists ladders $\widetilde{\lambda_1}, \widetilde{\lambda_2}$ such that $\Phi(\widetilde{\lambda_i}) = \Phi(\lambda_i)$ and $\widetilde{\lambda_2} \circ \widetilde{\lambda_1}$ is defined.

Proof. Let a be the codomain of λ_1 and b be the domain of λ_2. By hypothesis, $\Phi(a) = \Phi(b)$ (the problem is that there may be a different number of 0-labeled edges between nonzero labels of a and b, and a different number of primes on the edges). The proof strategy will be as follows: first, permute the labels of a and of b so that all instances of 0 are to the right of all nonzero labels; second, add additional 0 uprights to the right of a ladder so that the two ladders have the same number of 0 labels; finally, add primes to uprights as needed so that we may compose the ladders. We are finished when we note that none of the above moves change the ladder's image under Φ.

First, we compose with webs which permute the labels of a to be of the form

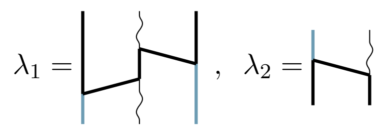
$$\widetilde{a} = x_1^{(p_1)} \dots x_k^{(p_k)} 0^{(q_1)} \dots 0^{(q_r)}$$

where each $x_i \neq 0$. To this end, suppose we have a zero label to the left of a nonzero label. Then the rung

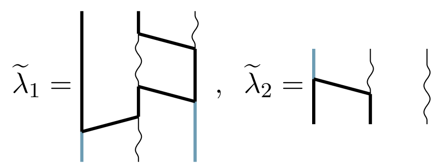


switches the order of the zero and nonzero label. Composing several of these together permutes the labels of a to be $\widetilde{a}$; call this ladder $\widehat{\lambda}_1$. Similarly, repeat the above procedure "upside-down" to write b in the form $\widetilde{b} = x_1^{(p'_1)} \dots x_k^{(p'_k)} 0^{(q'_1)} \dots 0^{(q'_s)}$, calling the new ladder $\widehat{\lambda}_2$. Now note that $\widetilde{a}$ *almost* equals $\widetilde{b}$ (so we are almost done), except that we could have $r \neq s$ and some $p_i \neq p'_i$, $q_j \neq q'_j$. However, these problems are easily dealt with. Suppose $r > s$, the case when $r < s$ can be treated similarly. Then tensor $\widehat{\lambda}_2$ with $r - s$ copies of uprights labeled 0 with no rungs attached. Finally, if we have some $p_i \neq p_i'$ (say $p_i > p_i'$), add $p_i - p_i'$ primes to the entire i^{th} upright of $\widehat{\lambda}_2$. Call these resulting ladders $\widetilde{\lambda}_1, \widetilde{\lambda}_2$ Finally, $\widetilde{\lambda}_2 \circ \widetilde{\lambda}_1$ is defined and, since none of the above moves change the image under Φ, we have $\Phi(\widetilde{\lambda}_i) = \Phi(\lambda_i)$. □

Example: Suppose we have λ_1, λ_2 as below.



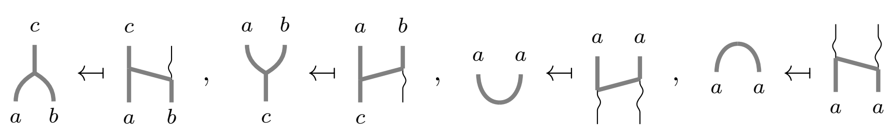
(we are omitting the labels of how many primes each edge carries). Then $\lambda_2 \circ \lambda_1$ is not defined, but $\Phi(\lambda_2) \circ \Phi(\lambda_1)$ is defined. Applying Lemma 6.0.9 to λ_1, λ_2, taking the ladders



we may now compose $\widetilde{\lambda}_2 \circ \widetilde{\lambda}_1$ and $\Phi(\widetilde{\lambda}_i) = \Phi(\lambda_i)$.

Corollary 6.0.10. Let w be a web in $\mathbf{Web}(\mathfrak{sp}_6)$. Then there exists a ladder λ with $\Phi(\lambda) = w$. In other words, we may "ladderize" any web.

Proof. First, put w in "Morse position" with respect to the height, so that all vertices and horizontal tangents of w occur at distinct heights. Enumerate each vertex, cap, and cup from bottom to top. We may "ladderize" each vertex, cap, and cup via:



and vertical mirror images. Now apply Lemma 6.0.9 inductively, starting at the bottom, to obtain a ladder λ with $\Phi(\lambda) = w$. □

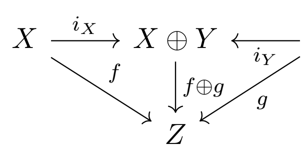
CHAPTER 7

Tr($\mathbf{Web}(\mathfrak{sp}_6)$)

As further evidence that we have an isomorphism $\mathbf{Web}(\mathfrak{sp}_6) \cong \mathbf{FundRep}(U_q(\mathfrak{sp}_6))$, we show that $\mathbf{Web}(\mathfrak{sp}_6)$ has the correct *categorical trace* [1]. This section will use some categorical constructions which we have not yet seen, so we provide some background for them here.

Definition 7.0.1. A *linear* category is a category in which each hom-set carries the structure of a vector space.

Given two objects X, Y, their *direct sum* (or *coproduct*), if it exists is an object $X \oplus Y$ equipped with morphisms $i_X : X \to X \oplus Y$, $i_Y : Y \to X \oplus Y$ such that for any object Z and morphisms $f : X \to Z$ and $g : Y \to Z$, there exists a unique morphism $f \oplus g : X \oplus Y \to Z$ such that the diagram



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commutes.

A category is *additive* if for all objects X, Y, their direct sum $X \oplus Y$ exists

A monoidal additive, linear category $\mathcal{C}$ is *semisimple* if

- all idempotents split, meaning if $e : A \to A$ is an idempotent, there exists an object B and morphisms $r : A \to B$ and $s : B \to A$ such that $s \circ r = e$ and $r \circ s = id_B$
- there exist *simple* objects $\{X_i\}_{i \in \mathcal{I}}$ such that $\mathrm{Hom}_{\mathcal{C}}(X_i, X_j) = \delta_{ij}\Bbbk$ (where $\Bbbk$ is the base field), and for any objects V, W, the composition map

$$\bigoplus_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(V, X_i) \otimes \mathrm{Hom}_{\mathcal{C}}(X_i, W) \to \mathrm{Hom}_{\mathcal{C}}(V, W)$$

  is an isomorphism.

As an example, given a simple $\mathfrak{g}$, $\mathbf{Rep}(U_q(\mathfrak{g}))$ is linear, because for any objects V, W, $\mathrm{Hom}(V, W)$

is a vector space; it is semisimple by Schur's Lemma 2.2.4, with simple objects the irreducible representations.

Given a category, we may wish to find a group to describe the category. We do so with the following construction.

Definition 7.0.2. Given an additive linear category $\mathcal{C}$, the *Grothendieck group* $K_0(\mathcal{C})$ is the free abelian group on isomorphism classes of $\mathcal{C}$ modulo the relation that $[X \oplus Y] = [X] + [Y]$.

As an example, $K_0(\mathbf{Rep}(U_q(\mathfrak{g})))$ is a free group on the irreducible representations of $U_q(\mathfrak{g})$ (modulo $[X \oplus Y] = [X] + [Y]$). Because $U_q(\mathfrak{g})$ is monoidal, we can define a product on $K_0(\mathbf{Rep}(U_q(\mathfrak{g})))$ by $[X] \times [Y] = [X \otimes Y]$. Doing so, we obtain the *representation ring* $R(U_q(\mathfrak{g}))$.

In the case that the category $\mathcal{C}$ is linear, there is another construction we may apply to obtain an abelian group from $\mathcal{C}$, which is analogous to the Grothendieck group, but can be easier to compute.

Definition 7.0.3. Let $\mathcal{C}$ be a linear category. The *categorical trace* (or zeroth Hochschild homology) of $\mathcal{C}$ is the vector space

$$\mathrm{Tr}(\mathcal{C}) := \bigoplus_{X \in Obj(\mathcal{C})} \mathrm{End}(X) \Big/ \langle fg - gf \rangle$$

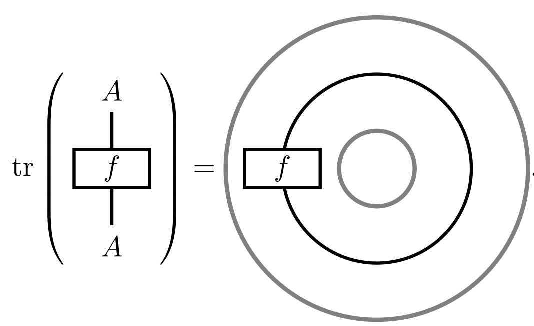
where f, g run through all morphisms $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$ for all $X, Y \in Obj(\mathcal{C})$.

Sometimes, we write $\mathrm{tr}(f)$ for the equivalence class of an endomorphism of f in $\mathrm{Tr}(\mathcal{C})$.

The categorical trace satisfies the following properties:

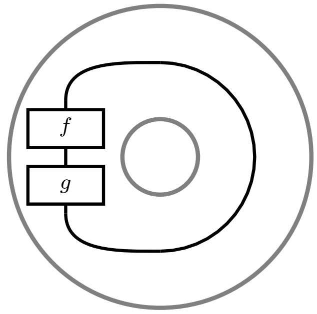
- $\mathrm{Tr}(\mathcal{C}) = \mathrm{Tr}(Kar(\mathcal{C}))$ where $Kar(\mathcal{C})$ denotes the Karoubi (or idempotent) completion of $\mathcal{C}$.
- If $\mathcal{C}$ is semisimple, then $K_0(\mathcal{C}) \xrightarrow{\sim} \mathrm{Tr}(\mathcal{C})$ via the "generalized Chern character" map $[X] \mapsto [id_X]$.

As monoidal categories admit diagrammatic descriptions, we expect the trace to, as well. Indeed, this is the case. Given an endomorphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, we depict $\mathrm{tr}(f)$ by putting f in an annulus and connecting the ends together; i.e.



We then have, for $f : A \to B$ and $g : B \to A$

$\mathrm{tr}(fg) =$



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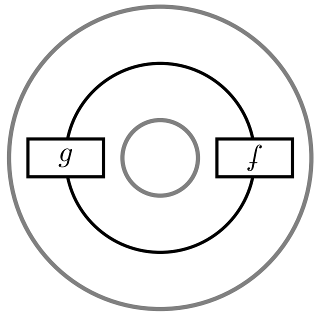

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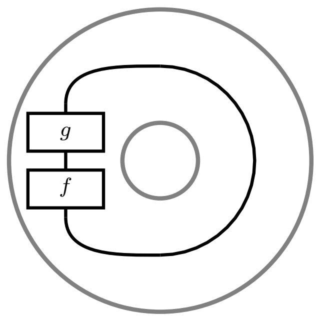


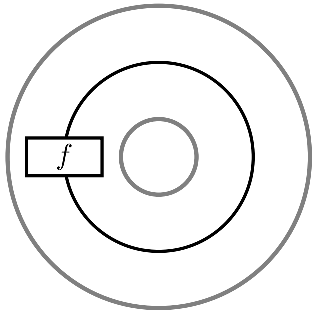
$= \mathrm{tr}(gf)$

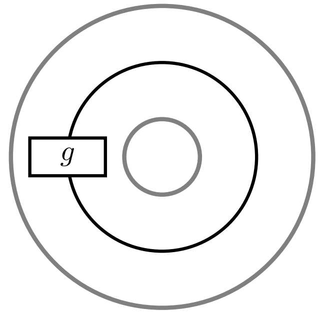
via an isotopy in the annulus.

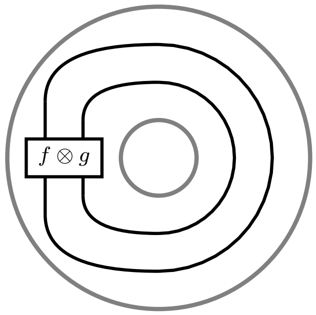
In the case that $\mathcal{C}$ is monoidal or braided monoidal, we can say a bit more about $\mathrm{Tr}(\mathcal{C})$.

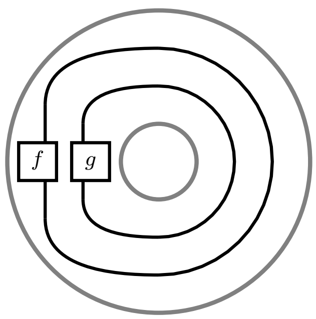
Lemma 7.0.4. When $\mathcal{C}$ is monoidal, $\mathrm{Tr}(\mathcal{C})$ inherits the structure of a unital associative algebra, with multiplication defined by $\mathrm{tr}(f) \cdot \mathrm{tr}(g) = \mathrm{tr}(f \otimes g)$. Further, if $\mathcal{C}$ is monoidal and braided, then $\mathrm{Tr}(\mathcal{C})$ is a commutative ring.

Proof. Note that the multiplication is given by

$\mathrm{tr}(f)\,\mathrm{tr}(g) =$ 

$\cdot$ 

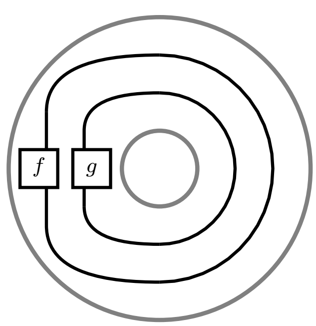
$=$ 

$=$ 

$= \mathrm{tr}(f \otimes g)$

from which it is clear that the unit is the identity morphism on the monoidal unit, and associativity follows by the associators of $\mathcal{C}$.

To see that $\mathrm{Tr}(\mathcal{C})$ is commutative, we see

$\mathrm{tr}(f) \cdot \mathrm{tr}(g) =$ 

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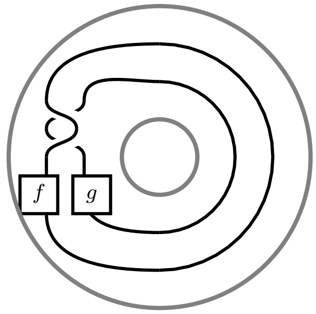

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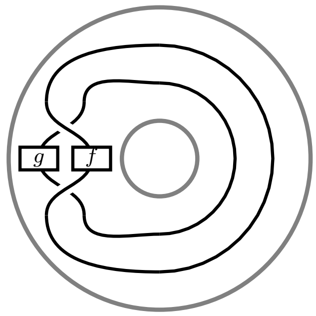

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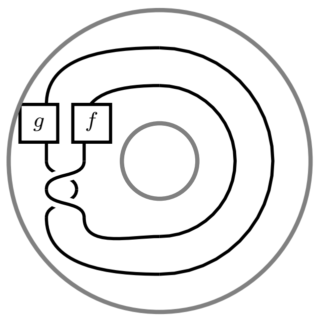

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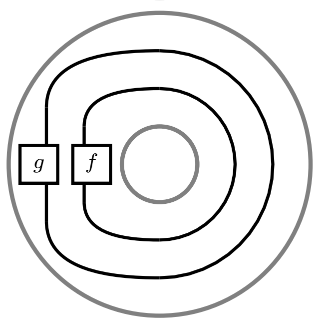


$= \mathrm{tr}(g) \cdot \mathrm{tr}(f)$

where we have applied the naturality of the braiding in the third line, and used the annular structure in the fourth. □

In the case of $\mathcal{C} = \mathbf{FundRep}(U_q(\mathfrak{g}))$, we have

$$\mathrm{Tr}(\mathbf{FundRep}(U_q(\mathfrak{g}))) = \mathrm{Tr}(\mathbf{Rep}(U_q(\mathfrak{g}))) \cong K_0(\mathbf{Rep}(U_q(\mathfrak{g}))) = R(U_q(\mathfrak{g})) = \mathbb{C}(q)[\Lambda_i]$$

where $R(U_q(\mathfrak{g}))$ is the representation ring and the Λ_i are the characters of the fundamental representations of $U_q(\mathfrak{g})$. Thus, if we can show that

$$\mathrm{Tr}(\mathbf{Web}(\mathfrak{sp}_6)) \cong \mathbb{C}(q)[\Lambda_1, \Lambda_2, \Lambda_3]$$

then we have arrived at the necessary condition that

$$\mathrm{Tr}(\mathbf{Web}(\mathfrak{sp}_6)) \cong \mathrm{Tr}(\mathbf{FundRep}(U_q(\mathfrak{sp}_6))).$$

We first make the following remark which relates $\mathrm{Tr}(\mathbf{Web}(\mathfrak{sp}_6))$ and $\mathrm{Tr}(\mathbf{Lad}(\mathfrak{sp}_6))$:

Remark 7.0.5. Recall from Definition 6.0.7 that there is a functor $\Phi : \mathbf{Lad}(\mathfrak{sp}_6) \to \mathbf{Web}(\mathfrak{sp}_6)$ which forgets the ladder structure and 0 edges. Then there is a map $\mathrm{Tr}(\Phi) : \mathrm{Tr}(\mathbf{Lad}(\mathfrak{sp}_6)) \to \mathrm{Tr}(\mathbf{Web}(\mathfrak{sp}_6))$ given by $\mathrm{Tr}(\Phi)(\mathrm{tr}(\lambda)) = \mathrm{tr}(\Phi(\lambda))$, and the diagram

$$\begin{array}{ccc} \mathbf{Lad}(\mathfrak{sp}_6) & \xrightarrow{\Phi} & \mathbf{Web}(\mathfrak{sp}_6) \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ \mathrm{Tr}(\mathbf{Lad}(\mathfrak{sp}_6)) & \xrightarrow{\mathrm{Tr}(\Phi)} & \mathrm{Tr}(\mathbf{Web}(\mathfrak{sp}_6)) \end{array}$$

commutes. In other words, if w is a web and λ is a ladder with $\Phi(\lambda) = w$, then $\mathrm{Tr}(\Phi)(\mathrm{tr}(\lambda)) = \mathrm{tr}(w)$.

Theorem 7.0.6. There is an isomorphism of algebras

$$\tau : \mathbb{C}(q)[\Lambda_1, \Lambda_2, \Lambda_3] \to \mathrm{Tr}(\mathbf{Web}(\mathfrak{sp}_6))$$

given by sending $\tau(\Lambda_i) = \mathrm{tr}(id_i)$

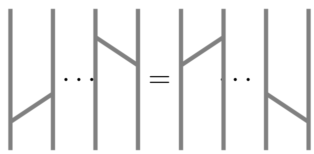
Proof. First, we show that τ is surjective. The proof follows [23]. Let w be a web in $\mathbf{Web}(\mathfrak{sp}_6)$ (with the same domain and codomain), and apply Corollary 6.0.10 to find a laddder λ with $\Phi(\lambda) = w$. Suppose λ has m uprights. The idea is to use a partial order (in this case, that of the $\mathfrak{sl}_{m-1}$ weight lattice) to simplify λ.

Between each pair of rung, we may take a slice through λ and define the tuple $k = (k_1, \ldots, k_m)$, where k_i is the mass of the i^{th} upright (recall the mass of an upright labeled by $x^{(i)}$ is $\mu\left(x^{(i)}\right) = x+2i$). Doing this between every pair of rungs gives a cyclic sequence $w_1, \ldots, w_r$ of such tuples. We interpret these sequences as $\mathfrak{gl}_m$ weights. We transition to $\mathfrak{sl}_m$ weights via

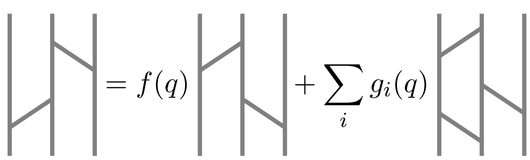
$$k = (k_1, \dots, k_{m+1}) \mapsto (k_1 - k_2, \dots, k_m - k_{m+1}) = \widetilde{k}.$$

Thus, we have defined a cyclic sequence $\widetilde{k}_1, \dots, \widetilde{k}_r$ of $\mathfrak{sl}_m$ weights for our ladder. With respect to the standard partial order on the $\mathfrak{sl}_{m-1}$ weight lattice, one of these weights is minimal. Being minimal, the weight must occur after an F-rung and before an E-rung (reading bottom to top). Suppose the F-rung connects the $i^{th}, (i+1)^{st}$ uprights, and the F-rung connects the $j^{th}, (j+1)^{st}$ uprights.

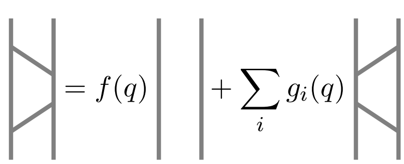
We have a few cases. If $|i - j| > 1$, then we may use the "far commutivity relation"



(when $i < j$, there is a similar relation to use when $i > j$) to switch the order of the F-rung and the E-rung, thus raising (this instance) of the minimal weight. Thus, we either have higher minimal weight, or the minimal weight occurring fewer times. If $|i - j| = 1$, then we may use the rung swap relation



(and a similar relation to use if $i > j$) to write the ladder in terms of ladders with Es coming before F, again raising (this instance of) minimal weight. For the remainder of the argument, we assume the rungs all carry a label of $1^{(0)}$ or $0'$. If the rung has any other label, we may apply one of the "thick rung" relations to write it in terms of ladders with the rung in question having a smaller mass. In the case of $i = j$, we apply a "square relation"



to again write λ in terms of ladders with higher minimal weight.

Recall that in our ladder category, all relations preserve the mass of a slice. Thus, while the above moves raise minimal weight, the total mass is constant. Thus, the minimal weight is bounded above. Thus, we are eventually left with the cyclic sequence of $\mathfrak{sl}_{m-1}$ weights having length 1, meaning the ladder consists of concentric circles.

Thus, the trace of any ladder can be written as a $\mathbb{C}(q)$-linear sum of concentric circles (i.e., the trace of identity morphisms), so τ is surjective. Applying Remark 7.0.5, we have that $\operatorname{tr}(w)$ is also a $\mathbb{C}(q)$-linear sum of concentric circles.

We now show that τ is injective. Note that we have the commutative diagram

$$\begin{array}{ccc} \operatorname{Tr}(\mathbf{Web}(\mathfrak{sp}_6)) & \xrightarrow{\operatorname{Tr}(\Psi)} & \operatorname{Tr}(\mathbf{FundRep}(U_q(\mathfrak{sp}_6))) \\ \tau \uparrow & & \downarrow \cong \\ \mathbb{C}(q)[\Lambda_1, \Lambda_2, \Lambda_3] & \dashrightarrow & K_0(\mathbf{Rep}(U_q(\mathfrak{sp}_6))) \end{array}$$

which sends

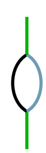
$$\begin{array}{ccc} \operatorname{tr}(id_i) & \longmapsto & \operatorname{tr}(id_{V_{\omega_i}}) \\ \uparrow & & \downarrow \\ \Lambda_i & & [V_{\omega_i}] \end{array}$$

so the composite map $\alpha : \mathbb{C}(q)[\Lambda_1, \Lambda_2, \Lambda_3] \to K_0(\mathbf{Rep}(U_q(\mathfrak{sp}_6)))$ is given by $\alpha(\Lambda_i) = [V_{\omega_i}]$. We claim α is injective (c.f. [9]). Indeed, if $\alpha\left(\sum_{i=i}^{N} f_i(q)\Lambda_1^{k_{1,i}}\Lambda_2^{k_{2,i}}\Lambda_3^{k_{3,i}}\right) = 0$, because the $\mathfrak{sp}_6$ weight lattice is partially ordered, some term has highest weight. But then the coefficient of this term must be 0, now induct on the number of terms.

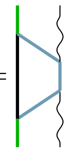
Thus, because α is injective, τ must be as well. $\square$

Examples: Here, we provide some examples to help elucidate the argument of Theorem 7.0.6.

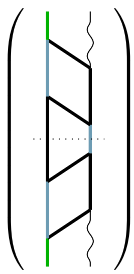
As a straightforward example (chosen in part to illustrate the reduction to all rungs having a label of 1), we apply the algorithm above reduce the trace of the $\operatorname{tr}(\mathbf{Web}(\mathfrak{sp}_6))$ web

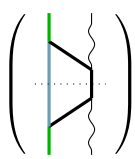
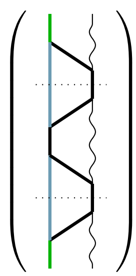
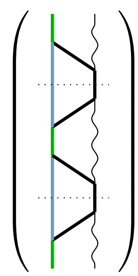
$w =$  .

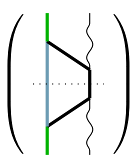
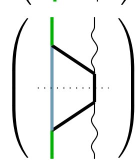
Of course, we expect the value after evaluating circles to be $(-[3]^2)\left(-\frac{[6][7][8]}{[2][3][4]}\right) = \frac{[3][6][7][8]}{[2][4]}$. First, we pull w back to a ladder, perhapst lifting it in the most naive way to

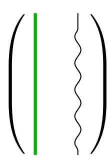
$\widetilde{w} =$ 

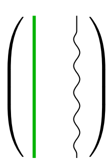
to $\mathbb{C}(q)$-linear sum of traces of identity strands. We have picked this example to demonstrate the reduction to the case that rungs have a label of 1. Throughout, we indicate the slice (or slices) of $\mathrm{tr}(\widetilde{w})$ with the lowest weight via a dotted line.

$\mathrm{tr}\,(\,\widetilde{w}\,) = \frac{1}{[2]^2[3]^2}\,\mathrm{tr}$ 

$= \frac{1}{[2]^2}\,\mathrm{tr}$  $+ \frac{1}{[2]^3[3]^2}\,\mathrm{tr}$  $- \frac{1}{[2]^2[3]}\,\mathrm{tr}$ 

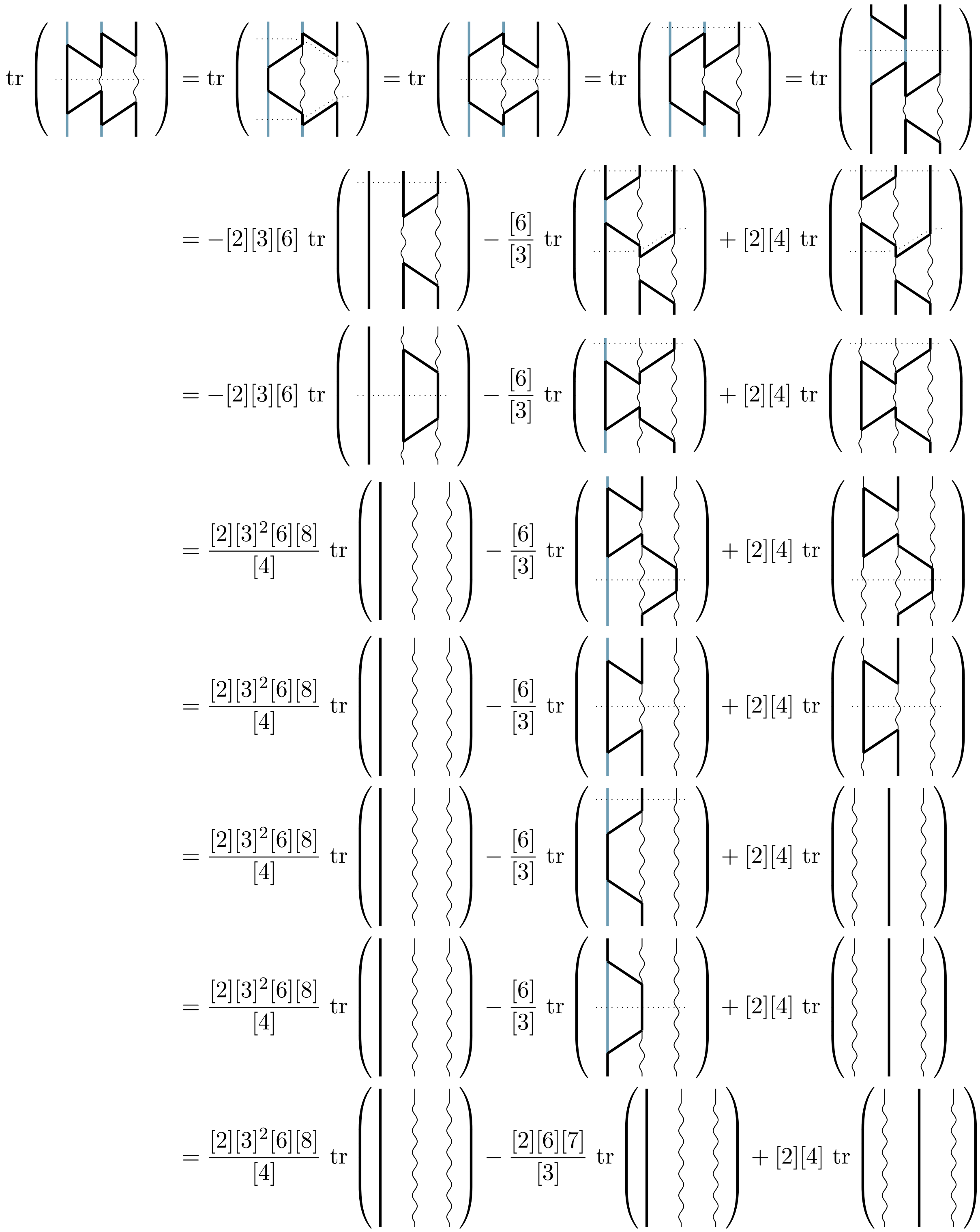
$= \frac{1}{[2]^2}\,\mathrm{tr}$  $+ 0 + \frac{[3]}{[2]^2}\,\mathrm{tr}$ 

$= \left(\frac{1+[3]}{[2]^2}\right)(-[3]^2)\,\mathrm{tr}$ 

$= -[3]^2\,\mathrm{tr}$  .

Looking back at w, of course we expect to just evaluate the digon and be left with the trace of id_3.

As another (less straightforward, but still not too difficult) example, chosen to illustrate relations involving rungs sharing an upright and taking advantage of the annular structure. Note that, to save space, we are displaying the first several equalities on the first line, and we are pushing some rungs closer than strictly allowed. We mark each minimal weight slice with a dotted line:



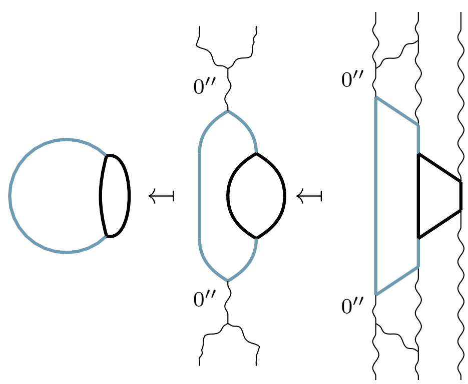
CHAPTER 8

$\mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_6)} = \mathbb{C}(q)$

Consider a closed web w; that is, a web with domain and codomain the empty sequence. Under Ψ, the empty sequence maps to the trivial representation, so $\Psi(w) \in \mathrm{End}_{\mathbf{FundRep}(U_q(\mathfrak{sp}_6))}(\mathbb{C}(q)) \cong \mathbb{C}(q)$. Thus, we expect relations in $\mathbf{Web}(\mathfrak{sp}_6)$ to evaluate w to an element of $\mathbb{C}(q)$. Indeed, we have

Theorem 8.0.1. We have $\mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_6)}(\emptyset) = \mathbb{C}(q)$.

Proof. Let $w \in \mathbf{Web}_{\mathfrak{sp}_6}(\emptyset)$. Apply Corollary 6.1.5 to find a ladder $\widetilde{w}$ with $\Phi(\widetilde{w}) = w$. Under Φ, the objects $0^{(k)}$ map to the empty object of $\mathbf{Web}(\mathfrak{sp}_6)$, so the domain and codomain of $\widetilde{w}$ is some sequence of 0 with some primes. Suppose further that $\widetilde{w} \in \mathrm{End}_{\mathbf{Lad}'}(0'^{\otimes k} \otimes 0^{\otimes \ell})$. For example, one may have

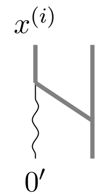


(and running the argument in this case may be illustrative; one should expect a value of $-\frac{[2][3][7][8]}{[4]}$).

Suppose further that k is *minimal*; the argument proceeds by induction on k. First, suppose $k = 0$. Since $\widetilde{w} \in \mathrm{End}_{\mathbf{Lad}'(\mathfrak{sp}_6)}(0^{\otimes \ell})$, there can be no rungs, so $\widetilde{w}$ is a multiple of $id_{0^{\otimes \ell}}$, which maps to a scalar under the map to $\mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_{2n})}$.

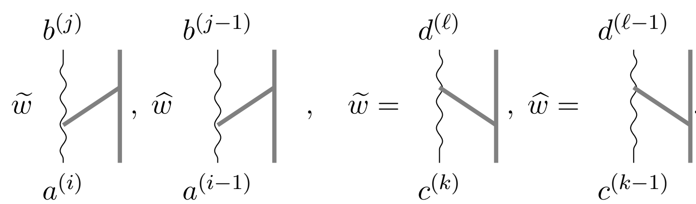
Now, suppose $\widetilde{w} \in \mathrm{End}_{\mathbf{Lad}'}(0'^{\otimes k} \otimes 0^{\otimes \ell})$. We show that we may write $\widetilde{w} = \sum_i f_i(q)\widetilde{w}_i$, where for each i, there exists $\widetilde{\widetilde{w}}_i \in \mathrm{Hom}_{\mathbf{Lad}(\mathfrak{sp}_6)}(0 \otimes 0'^{\otimes k-1} \otimes 0^{\otimes \ell})$ with $\Phi(\widetilde{\widetilde{w}}_i) = \Phi(\widetilde{w}_i)$, from which the induction is complete

As in the proof of Theorem 7.0.6, we write $\widetilde{w}$ as a $\mathbb{C}(q)$-linear combination of webs which each have all E rungs below all F rungs. Consider the leftmost upright, we claim that the *entire* leftmost upright has at least one prime. Indeed, if it has any rungs attached to it, it first has type E rungs followed by type F rungs. Consider the first type E rung. It merges an edge with the primed edge $0'$, but any merge with a primed edge results in a primed edge; *i.e.* , the bottom-left rung of our ladder has the form



and any allowable edge labels for $x^{(i)}$ have $i > 0$. In fact, this is true if we replace the $0'$ on the bottom with any primed edge. By induction, the leftmost edge is primed after applying all the type E rungs. Applying this argument "upside-down" and noting that we end with the primed edge $0'$, we see that the leftmost edge before all the type F rungs is also primed. Thus, every label on the leftmost upright is primed.

Now, for each i, pick $\widetilde{\widetilde{w}}_i$ to be the same as $\widetilde{w}_i$, but with the entire leftmost upright with one fewer prime. That is, on all the leftmost rungs, we replace the rungs as follows



Note that this is well-defined because the leftmost upright is all primed. Now we have written $w = \sum_i f_i(q)\widetilde{\widetilde{w}}_i$ where each $\widetilde{\widetilde{w}}_i \in \mathrm{Hom}_{\mathbf{Lad}(\mathfrak{sp}_6)}(0 \otimes 0'^{\otimes(k-1)} \otimes 0 \otimes \ell)$. From here, we may braid the leftmost 0 edge past all the $0'$ edges to write w as a $\mathbb{C}(q)$-linear combintation of elements of $\mathrm{Hom}_{\mathbf{Lad}(\mathfrak{sp}_6)}(0'^{\otimes(k-1)} \otimes 0^{\otimes(\ell+1)})$; the result now follows by induction on k.

We have thus shown that $\dim \mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_6)}(\emptyset) \leq 1$. It remains to show $\dim \mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_6)}(\emptyset) \geq 1$. Indeed, recall from Theorem 5.0.1 that $\Psi : \mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{FundRep}(U_q(\mathfrak{sp}_6))$ is *full.* Thus, Ψ gives a surjective map $\mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_6)}(\emptyset \to \mathrm{End}_{\mathbf{FundRep}(U_q(\mathfrak{sp}_6))}(\mathbb{C}(q))$. But $\mathrm{End}_{\mathbf{FundRep}(U_q(\mathfrak{sp}_6))}(\mathbb{C}(q)) \cong \mathbb{C}(q)$ by Schur's Lemma 2.2.4, so $\dim \mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_6)}(\emptyset) \geq 1$. Thus, $\dim \mathrm{End}_{\mathbf{Web}(\mathfrak{sp}_6)}(\emptyset) = 1$. □

CHAPTER 9

Branching Functors and Webs

In [19], Morrison finds a conjectural presentation for the web category $\mathbf{Web}(\mathfrak{sl}_n)$ (which was later proved to be correct in [4]. One of his main tools was to use the inclusions $U_q(\mathfrak{sl}_{n-1}) \hookrightarrow U_q(\mathfrak{sl}_n)$ to inductively find $\mathbf{Web}(\mathfrak{sl}_n)$ relations from $\mathbf{Web}(\mathfrak{sl}_{n-1})$ relations. While the equivalence between $\mathbf{Web}(\mathfrak{sl}_n)$ and $\mathbf{FundRep}(U_q(\mathfrak{sl}_n))$ was not proved until [4], the relations Morrison found using this approach were sufficient to eventually prove faithfulness of $\mathbf{Web}(\mathfrak{sl}_n) \to \mathbf{FundRep}(U_q(\mathfrak{sl}_n))$. We apply the same strategy here, though we shall see (and as Morrison notes in [19]), the restriction maps $\mathbf{FundRep}(U_q(\mathfrak{sp}_{2n})) \to \mathbf{FundRep}(U_q(\mathfrak{sp}_{2(n-1)}))$ have an added difficulty that is absent in the Type A case.

Because the restriction of irreducible representations map to direct sums of irreducible representations, we will need "expand" our category slightly.

Definition 9.0.1. Let $\mathcal{C}$ be a linear category (that is, one in which the Hom sets are vector spaces). Define the *additive closure* of $\mathcal{C}$, denoted $\mathcal{C}^{\oplus}$, to be a category whose objects are formal finite direct sums of objects of $\mathcal{C}$, and whose morphisms are matrices of morphisms of $\mathcal{C}$. The composition of morphisms in $\mathcal{C}^{\oplus}$ is the familiar matrix multiplication (which we may do, because $\mathcal{C}$ is linear).

If $U_q(\mathfrak{g}')$ is a subalgebra of $U_q(\mathfrak{g})$, the inclusion map induces a restriction functor $r : \mathbf{Rep}(U_q(\mathfrak{g})) \to \mathbf{Rep}(U_q(\mathfrak{g}'))$. If r actually maps fundamental representations to (direct sums of tensor products of) fundamental representations, we obtain a functor $r : \mathbf{FundRep}(U_q(\mathfrak{g})) \to \mathbf{FundRep}(U_q(\mathfrak{g}'))^{\oplus}$. Now suppose we further have an equivalence of categories $\mathbf{FundRep}(U_q(\mathfrak{g}')) \xrightarrow{\sim} \mathbf{Web}(\mathfrak{g}')$ and a full, essentially surjective functor $\Psi : \mathbf{FundRep}(U_q(\mathfrak{g})) \to \mathbf{Web}(\mathfrak{g})$. We thus expect a map $\widetilde{r} : \mathbf{Web}(\mathfrak{g}) \to \mathbf{Web}(\mathfrak{g}')^{\oplus}$ such that

$$\begin{array}{ccc} \mathbf{Web}(\mathfrak{g}) & \xrightarrow{\quad\widetilde{r}\quad} & \mathbf{Web}(\mathfrak{g}')^{\oplus} \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\cong} \\ \mathbf{FundRep}(U_q(\mathfrak{g})) & \xrightarrow{\ r\ } & \mathbf{FundRep}(U_q(\mathfrak{g}'))^{\oplus} \end{array}$$

commutes.

9.1 Two functors $\mathbf{Web}(\mathfrak{sp}_4) \to \mathbf{Web}(\mathfrak{sl}_2)^{\oplus}$

Recall that $U_q(\mathfrak{sp}_4)$ and $U_q(\mathfrak{sl}_2)$ have Cartan matrices

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \quad , \quad \begin{pmatrix} 2 \end{pmatrix}$$

respectively. Recalling the definition of a quantum group from a Cartan matrix, this tells us the relations between the generators E_1, F_1, K_1, K_1^{-1} are the same as those of $U_q(\mathfrak{sl}_2)$, and similarly for E_2, F_2, K_2, K_2^{-1}. Thus, we have inclusions $i_L : U_q(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{sp}_4)$ and $i_R : U_q(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{sp}_4)$; given by remembering only the action of generators of $U_q(\mathfrak{sp}_4)$ indexed by 1 and 2, respectively. Restricting these inclusions give maps $\phi_L, \phi_R : \mathbf{Rep}(U_q(\mathfrak{sp}_4)) \to \mathbf{Rep}(U_q(\mathfrak{sl}_2))^{\oplus}$. A computation shows that the image of a fundamental $U_q(\mathfrak{sp}_4)$ is a fundamental $U_q(\mathfrak{sl}_2)$ representation, so ϕ_L, ϕ_R actually map $\mathbf{FundRep}(U_q(\mathfrak{sp}_4)) \to \mathbf{FundRep}(U_q(\mathfrak{sl}_2))^{\oplus}$. Thus, we expect maps $\widetilde{\phi}_L, \widetilde{\phi}_R : \mathbf{Web}(\mathfrak{sp}_4) \to \mathbf{Web}(\mathfrak{sl}_2)^{\oplus}$.

Theorem 9.1.1. There are functors $\widetilde{\phi}_L, \widetilde{\phi}_R : \mathbf{Web}(\mathfrak{sp}_4) \to \mathbf{Web}(\mathfrak{sl}_2)^{\oplus}$ such that

$$\begin{array}{ccc} \mathbf{Web}(\mathfrak{sp}_4) & \xrightarrow{\quad \widetilde{\phi} \quad} & \mathbf{Web}(\mathfrak{sl}_2)^{\oplus} \\ \downarrow \Psi & & \downarrow \cong \\ \mathbf{FundRep}(U_q(\mathfrak{sp}_4)) & \xrightarrow{\phi_L} & \mathbf{FundRep}(U_q(\mathfrak{sl}_2))^{\oplus} \end{array}$$

commutes.

Proof. The proof is by construction. First, let's examine ϕ_R. Forgetting the generators E_2, F_2, K_2, K_2^{-1} of $U_q(\mathfrak{sp}_4)$, we see that the representations V, W decompose as

$$\phi_R(\mathbb{C}(q)) = \Bbbk$$

$$\phi_R(V) = \langle x_1, x_2 \rangle \oplus \langle x_3, x_4 \rangle = V \oplus V$$

$$\phi_R(W) = \langle v_{12} \rangle \oplus \langle v_{13}, v_0, v_{24} \rangle \oplus \langle v_{34} \rangle = \mathbb{C}(q) \oplus (V_2 \oplus \mathbb{C}(q)) \cong \mathbb{C}(q) \oplus (V \otimes V)$$

where in the last line above, we identify $V_2 \oplus \mathbb{C}(q) \cong V \otimes V$; we pick the first copy of $\mathbb{C}(q)$ to be generated by v_{12} and the copy of $\mathbb{C}(q)$ living in $V \otimes V$ to be generated by v_{34}. (Recalling the definitions of the representations in the background, we had $x_3 = F_2(x_2)$, but we are forgetting about the action of F_2; similarly for the others).

We use the notation $v_1 = v^+$, $v_2 = v^-$, $v_3 = w^+$, $v_4 = w^-$, because v_1, v_2 are the vectors of weight $+1, -1$ of the first copy of V in $\phi_R(V)$, and similarly v_3, v_4 are the vectors of $\mathfrak{sl}_2$ weight $+1, -1$ of the second copy of V in $\Phi_L(V)$. Similarly, we will use the notation $w_{12} = c$, $w_{13} = v_{++}$, $w_0 = v_0$, $w_{24} = v_{--}$, and $w_{34} = c_2$.

Using this notation, our maps between $U_q(\mathfrak{sp}_4)$ representations are given by

$$\begin{aligned} i_{\Bbbk} : 1 \mapsto\ & q^2 v^+ \otimes w^- - q v^- \otimes w^+ \\ & -q^{-2} w^- \otimes v^+ + q^{-1} w^+ \otimes v^- \end{aligned}$$

$$i_W : \begin{cases} w_{12} \mapsto q v^+ \otimes v^- - v^- \otimes v^+ \\ w_{13} \mapsto q v^+ \otimes w^+ - w^+ \otimes v^+ \\ w_{14} \mapsto q v^+ \otimes w^- - w^- \otimes v^+ \\ w_{23} \mapsto q v^- \otimes w^+ - w^+ \otimes v^- \\ w_{24} \mapsto q v^- \otimes w^- - w^- \otimes v^- \end{cases}$$

$$p_{\Bbbk} : \begin{cases} v^+ \otimes w^- \mapsto -q^2 \\ v^- \otimes w^+ \mapsto q \\ w^+ \otimes v^- \mapsto -q^{-1} \\ w^- \otimes v^+ \mapsto q^{-2} \end{cases}$$

$$p_W : \begin{cases} v^+ \otimes v^- \mapsto -[2]c \\ v^+ \otimes w^+ \mapsto -[2]v_{++} \\ v^- \otimes w^- \mapsto -[2]v_{--} \\ w^+ \otimes w^- \mapsto -[2]c_2 \\ v^- \otimes v^+ \mapsto q^{-1}[2]c \\ w^+ \otimes v^+ \mapsto q^{-1}[2]v_{++} \\ w^- \otimes v^- \mapsto q^{-1}[2]v_{--} \\ w^- \otimes w^+ \mapsto q^{-1}[2]c_2 \\ v^+ \otimes w^- \mapsto -w_0 \\ v^- \otimes w^+ \mapsto -q w_0 \\ w^+ \otimes v^- \mapsto q^{-1} w_0 \\ w^- \otimes v^+ \mapsto w_0. \end{cases}$$

We compare these to these the maps $i : \Bbbk \to V \otimes V$ and $p : V \otimes V \to \Bbbk$ between $U_q(\mathfrak{sl}_2)$ given by

$$p : \begin{cases} v_+ \otimes v_+ & \mapsto 0 \\ v_+ \otimes v_- & \mapsto -1 \\ v_- \otimes v_+ & \mapsto q^{-1} \\ v_- \otimes v_- & \mapsto 0 \end{cases} \quad \text{and} \quad i : 1 \mapsto q v_+ \otimes v_- - v_- \otimes v_+.$$

Thus, one can check that ϕ_R sends

$$id_V \mapsto \begin{pmatrix} id_V & 0 \\ 0 & id_V \end{pmatrix}$$

$$id_W \mapsto \begin{pmatrix} id_{\mathbb{C}(q)} & 0 \\ 0 & id_{V\otimes V} \end{pmatrix}$$

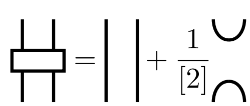
$$p_{\mathbb{C}(q)} \mapsto \begin{pmatrix} 0 & q^2 p & q^{-1}p & 0 \end{pmatrix}$$

$$p_W \mapsto \begin{pmatrix} [2]p & 0 & 0 & 0 \\ 0 & -([2]id_V + p) & q^{-1}([2]id_V + p) & -(i \circ p) \end{pmatrix}$$

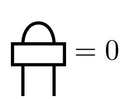
$$p_{\mathbb{C}(q)} \mapsto \begin{pmatrix} 0 \\ qp \\ q^{-2}p \\ 0 \end{pmatrix}$$

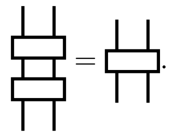
$$i_W \mapsto \begin{pmatrix} i_{\mathbb{C}(q)} & 0 \\ 0 & q\, id_{V\otimes V} - \frac{q}{[2]} i \circ p \\ 0 & - - id_{V\otimes V} + \frac{1}{[2]} i \circ p \\ 0 & i \circ p \end{pmatrix}$$

We see multiples of that maps $id_{V\otimes V} + \frac{1}{[2]} i \circ p$ a few times above. Indeed, this map is the projection onto and inclusion from the irreducible $U_q(\mathfrak{sl}_2)$ representation $\mathrm{Sym}^2_q(V)$, called the *Jones-Wenzl projector* (to see this, note that it is an idempotent and composing it with p or i gives 0). Diagrammatically, we depict this map with a box, so

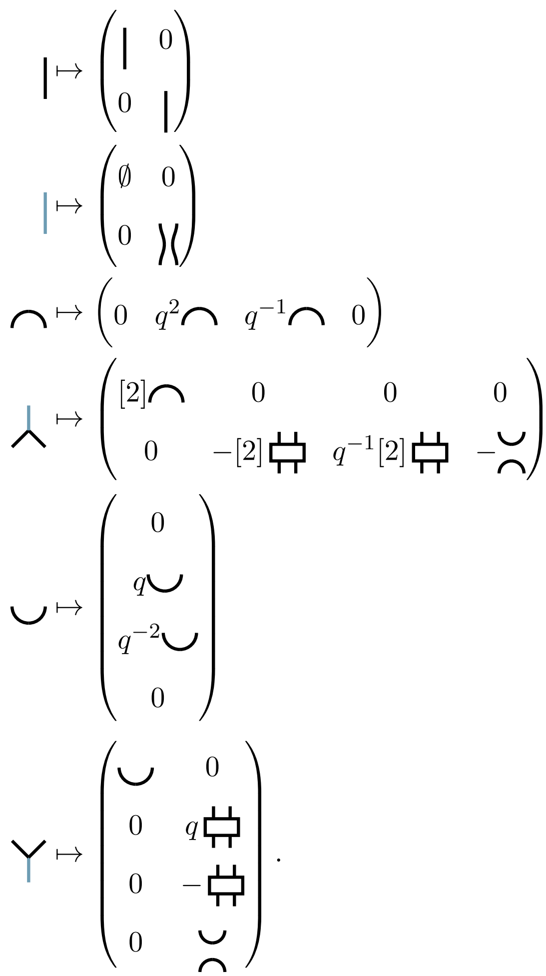


which then satisfies

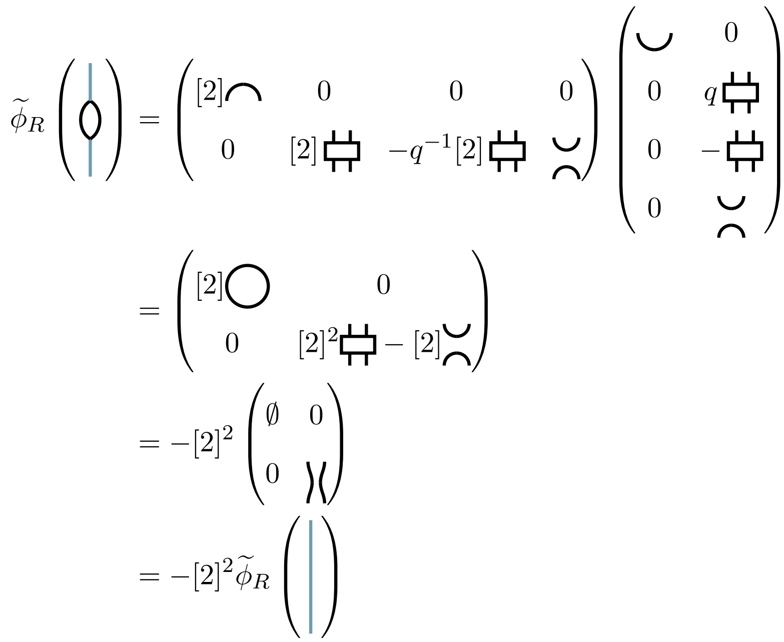




Thus, for the diagram in 9.1.1 to commute, we must define $\widetilde{\phi}_R$ by



We may check that $\widetilde{\phi}_R$ factors through the relations of $\mathbf{Web}(\mathfrak{sp}_4)$ [1] For example, we have



Now, consider ϕ_R. Under ϕ_R, we have

$$\begin{aligned}
\Bbbk &\mapsto \Bbbk \\
V &\mapsto \langle x_1 \rangle \oplus \langle x_2, x_3 \rangle \oplus \langle x_4 \rangle = \Bbbk \oplus V \oplus \Bbbk \\
W &\mapsto \langle v_{12}, v_{13} \rangle \oplus \langle v_0 \rangle \oplus \langle v_{24}, v_{34} \rangle = V \oplus \Bbbk \oplus V.
\end{aligned}$$

Now, the copy of $U_q(\mathfrak{sl}_2)$ sitting inside of $U_q(\mathfrak{sp}_4)$ generated by $E_2, F_2, K_2^{\pm}$ has roots twice as long (the second root of $U_q(\mathfrak{sp}_4)$ is long, $(\alpha_2, \alpha_2) = 4$). Thus, the gradings of the $U_q(\mathfrak{sl}_2)$ representations will all be doubled. Accordingly, we map into a re-scaled version of Temperly-Lieb, $\mathbf{TL}_{q^2}$. The

[1]Indeed, the original motivation for this section was to use these functors to find all web relations, before the strategy we used in Chapter 3 came to mind.

re-scaled $U_q(\mathfrak{sl}_2)$ module maps are given by

$$p : \begin{cases} v_+ \otimes v_+ & \mapsto 0 \\ v_+ \otimes v_- & \mapsto -q \\ v_- \otimes v_+ & \mapsto q^{-1} \\ v_- \otimes v_- & \mapsto 0 \end{cases} \quad \text{and} \quad i : 1 \mapsto qv_+ \otimes v_- - q^{-1} v_- \otimes v_+.$$

Applying the same method as above, tracing the $U_q(\mathfrak{sp}_4)$ maps through the identification ϕ_R, we have that

$$id_V \mapsto \begin{pmatrix} id_{\mathbb{C}(q)} & 0 & 0 \\ 0 & id_V & 0 \\ 0 & 0 & id_{\mathbb{C}(q)} \end{pmatrix}$$

$$id_W \mapsto \begin{pmatrix} id_V & 0 & 0 \\ 0 & id_{\mathbb{C}(q)} & 0 \\ 0 & 0 & id_V \end{pmatrix}$$

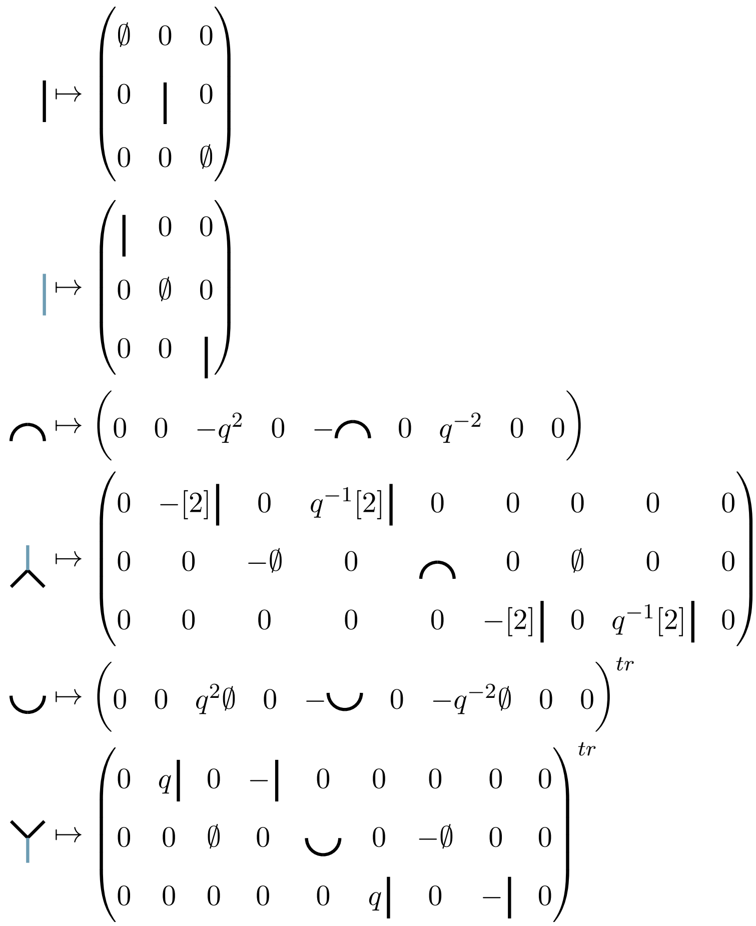
$$p_{\mathbb{C}(q)} \mapsto \begin{pmatrix} 0 & 0 & -q^2 id_{\mathbb{C}(q)} & 0 & -p & 0 & q^{-2} id_{\mathbb{C}(q)} & 0 & 0 \end{pmatrix}$$

$$p_W \mapsto \begin{pmatrix} 0 & -[2] id_V & 0 & q^{-1}[2] id_V & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -id_{\mathbb{C}(q)} & 0 & p & 0 & id_{\mathbb{C}(q)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -[2] id_V & 0 & q^{-1}[2] id_V & 0 \end{pmatrix}$$

$$i_{\mathbb{C}(q)} \mapsto \begin{pmatrix} 0 & 0 & q^2 id_{\mathbb{C}(q)} & 0 & -i & 0 & -q^{-2} id_{\mathbb{C}(q)} & 0 & 0 \end{pmatrix}^{tr}$$

$$i_W \mapsto \begin{pmatrix} 0 & q\, id_V & 0 & -id_V & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & id_{\mathbb{C}(q)} & 0 & i & 0 & -id_{\mathbb{C}(q)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q\, id_V & 0 & -\, id_V & 0 \end{pmatrix}^{tr}$$

Therefore, for the diagram in Theorem 9.1.1 to commute, we must define $\widetilde{\phi}_R$ by



Again, we can check that $\widetilde{\phi}_R$ factors through the $\mathbf{Web}(\mathfrak{sp}_4)$ relations. This is a very long process, so we omit doing so here, but checking them can be an enjoyable experience. □

9.2 A functor $\mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{Web}(\mathfrak{sp}_4)^{\oplus}$

We now consider the branching functors arising from the inclusion $U_q(\mathfrak{sp}_4) \hookrightarrow U_q(\mathfrak{sp}_6)$. First, one may be interested in remembering only the actions of $E_1, F_1, K_1^{\pm 1}, E_2, F_2, K_2^{\pm 1}$. In fact, these generate a copy of $U_q(\mathfrak{sl}_3)$ in $U_q(\mathfrak{sp}_6)$, so we have an in inclusion $U_q(\mathfrak{sl}_3) \hookrightarrow U_q(\mathfrak{sp}_6)$ and thus a map $\mathbf{FundRep}(U_q(\mathfrak{sp}_6)) \to \mathbf{Rep}(U_q(\mathfrak{sl}_3))$. If this map happens to map into $\mathbf{FundRep}(U_q(\mathfrak{sl}_3))^{\oplus}$, then we would have a functor $\mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{Web}(\mathfrak{sl}_3)$. However, under this map, $W \mapsto \Gamma_{1,0} \oplus \Gamma_{1,1} \oplus \Gamma_{0,1}$, and we cannot write this as the direct sum of tensor products of fundamentals (in $U_q(\mathfrak{sl}_3)$, we have $\Gamma_{1,0} \otimes \Gamma_{0,1} = \Gamma_{1,1} \oplus \Bbbk$, so we cannot put the copy of $\Gamma_{1,1}$ into a tensor product without having $\mathbb{k}$ as a summand). A similar problem happens when viewing the image of X. We could remedy this by instead mapping into $\mathbf{Kar}(\mathbf{Web}(U_q(\mathfrak{sl}_3)))$, but we did not puruse this. However, the actions of $E_2, F_2, K_2^{\pm 1}, E_3, F_3, K_3^{\pm 1}$ give an inclusion $U_q(\mathfrak{sp}_4) \hookrightarrow U_q(\mathfrak{sp}_6)$ and thus a map $\mathbf{FundRep}(U_q(\mathfrak{sp}_6)) \to \mathbf{Rep}(U_q(\mathfrak{sp}_4))$ which actually maps into $\mathbf{FundRep}(U_q(\mathfrak{sp}_4))^{\oplus}$. Thus, we expect the following theorem.

Theorem 9.2.1. There is a functor $\widetilde{\phi} : \mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{Web}(\mathfrak{sp}_4)^{\oplus}$ such that

$$\begin{array}{ccc} \mathbf{Web}(\mathfrak{sp}_6) & \xrightarrow{\widetilde{\phi}} & \mathbf{Web}(\mathfrak{sp}_4)^{\oplus} \\ \downarrow \Psi & & \downarrow \cong \\ \mathbf{FundRep}(U_q(\mathfrak{sp}_6)) & \xrightarrow{\phi} & \mathbf{FundRep}(U_q(\mathfrak{sp}_4))^{\oplus} \end{array}$$

commutes.

Proof. As in the proof of Theorem 9.1.1, the proof is by construction. The restriction map induced by including $U_q(\mathfrak{sp}_4) \hookrightarrow U_q(\mathfrak{sp}_6)$ is given by

$$\mathbb{k} \mapsto \mathbb{k}$$

$$V \mapsto \mathbb{k} \oplus V \oplus \mathbb{k}$$

$$W \mapsto V \oplus \mathbb{k} \oplus W \oplus V$$

$$X \mapsto W \oplus V \oplus W$$

We note that it will be most useful to idenntify the copy of $\mathbb{k}$ in W as being spanned by $[2]w_{0,1} - w_{0,2}$, as then this element includes into the copy of $V \otimes V$ in the easiest way.

On morphisms, the functor sends (to save space, we write $p_V : W \otimes V \to V$ for $(id_V \otimes p_{\mathbb{C}(q)}) \circ (i_W \otimes id_V)$ and $i_V : V \to W \otimes V$ for $(p_W \otimes id_V) \circ (id_V \otimes i_{\mathbb{C}(q)})$)

$$id_V \mapsto \begin{pmatrix} id_{\mathbb{C}(q)} & 0 & 0 \\ 0 & id_V & 0 \\ 0 & 0 & id_{\mathbb{C}(q)} \end{pmatrix}$$

$$id_W \mapsto \begin{pmatrix} id_V & 0 & 0 & 0 \\ 0 & id_{\mathbb{C}(q)} & 0 & 0 \\ 0 & 0 & id_W & 0 \\ 0 & 0 & 0 & id_V \end{pmatrix}$$

$$id_X \mapsto \begin{pmatrix} id_W & 0 & 0 \\ 0 & id_V & 0 \\ 0 & 0 & id_W \end{pmatrix}$$

$$p_{\mathbb{C}(q)} \mapsto \begin{pmatrix} 0 & 0 & -q^3\, id_{\mathbb{C}(q)} & 0 & -\, p_{\mathbb{C}(q)} & 0 & q^{-3}\, id_{\mathbb{C}(q)} & 0 & 0 \end{pmatrix}$$

$$p_W \mapsto \begin{pmatrix} 0 & -[3]\, id_V & 0 & q^{-1}[3]\, id_V & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -id_{\mathbb{C}(q)} & 0 & \frac{1}{[2]} id_{\mathbb{C}(q)} & 0 & id_{\mathbb{C}(q)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{[3]}{[2]} p_W & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -[3]\, id_V & 0 & q^{-1}[3]\, id_V & 0 \end{pmatrix}$$

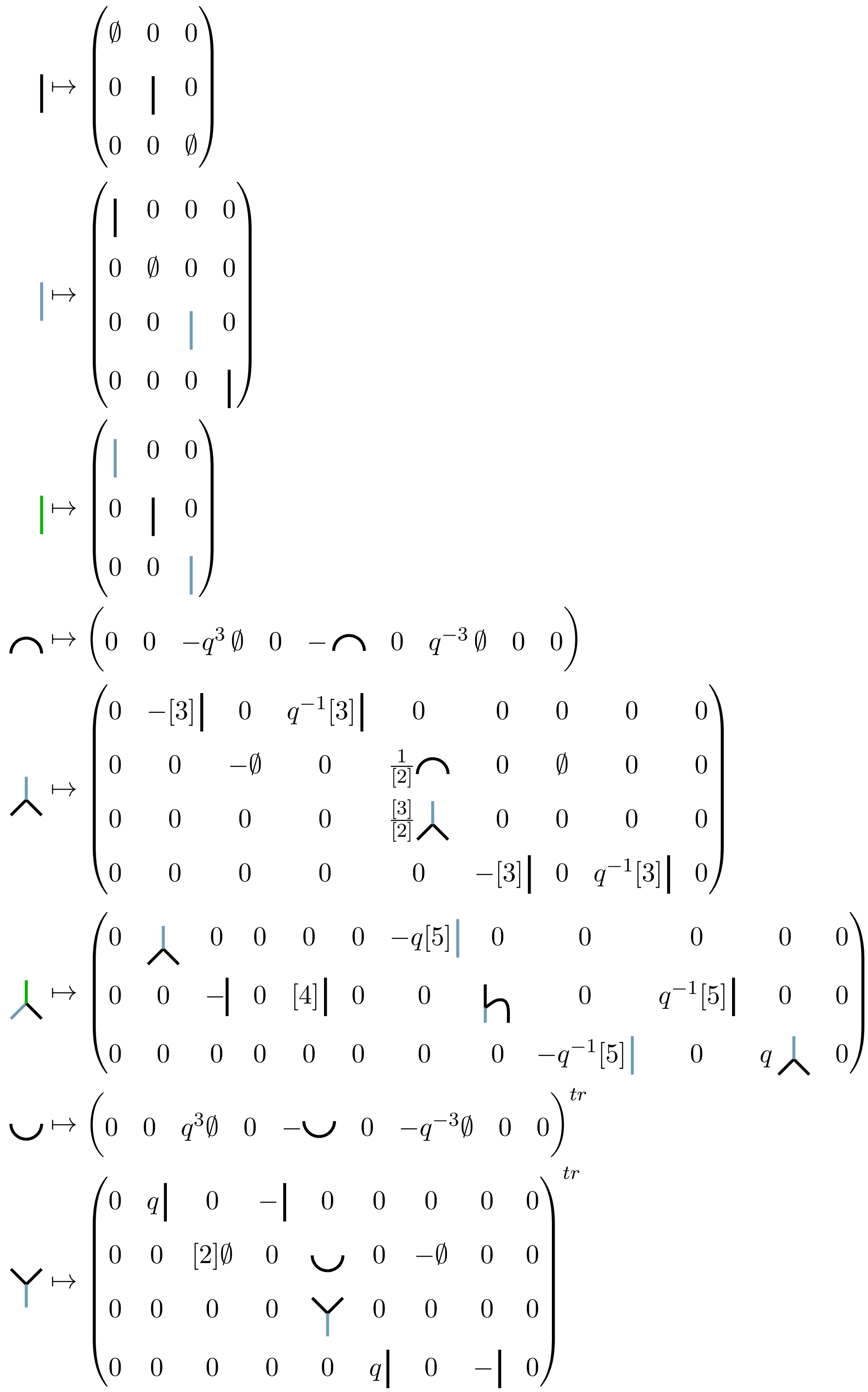
$$p_X \mapsto \begin{pmatrix} 0 & p_W & 0 & 0 & 0 & 0 & -q[5]\, id_W & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\, id_V & 0 & [4]\, id_V & 0 & 0 & p_V & 0 & q^{-1}[5]\, id_V & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1}[5]\, id_W & 0 & q\, p_W & 0 \end{pmatrix}$$

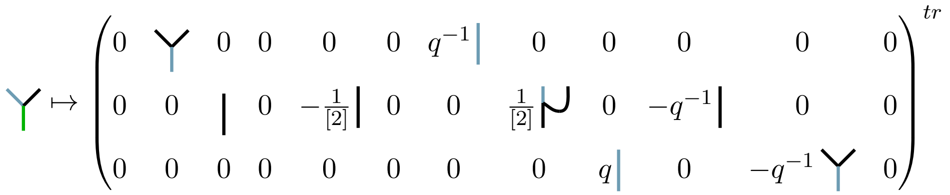
$$i_{\mathbb{C}(q)} \mapsto \begin{pmatrix} 0 & 0 & q^3 id_{\mathbb{C}(q)} & 0 & -i_{\mathbb{C}(q)} & 0 & -q^{-3} id_{\mathbb{C}(q)} & 0 & 0 \end{pmatrix}^{tr}$$

$$i_W \mapsto \begin{pmatrix} 0 & q\, id_V & 0 & -\, id_V & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [2] id_{\mathbb{C}(q)} & 0 & i_{\mathbb{C}(q)} & 0 & -id_{\mathbb{C}(q)} & 0 & 0 \\ 0 & 0 & 0 & 0 & i_W & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q\, id_V & 0 & -\, id_V & 0 \end{pmatrix}^{tr}$$

$$i_X \mapsto \begin{pmatrix} 0 & i_W & 0 & 0 & 0 & 0 & q^{-1}\, id_W & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & id_V & 0 & -\frac{1}{[2]}\, id_V & 0 & 0 & \frac{1}{[2]} p_V & 0 & -q^{-1}\, id_V & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q\, id_W & 0 & -q^{-1}\, i_W & 0 \end{pmatrix}^{tr}$$

Thus, in order for the diagram in Theorem 9.2.1 to commute, we must define $\widetilde{\phi}$ by





Checking that $\widetilde{\phi}$ factors through the relations of $\mathbf{Web}(\mathfrak{sp}_6)$ is even more intensive than before, but doing so is a good exercise (at least for "smaller relations," like digons). □

CHAPTER 10

Type C Link Invariants

While conjecture 3.0.6 remains open, we emphasize that we may use the braiding of $\mathbf{Web}(\mathfrak{sp}_6)$ in Equation 3.1.3 to compute link invariants. Given an oriented link diagram, we use Equation 3.1.3 to write the link diagram in terms of closed webs. We may then use Theorem 8.0.1 to evaluate the web, being left with an element of $\mathbb{C}(q)$.

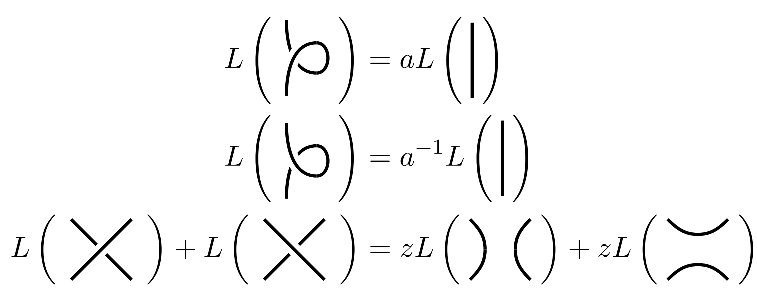
The Type C link invariant has been described before. In [20], Murakami and Ohtsuki give a formulation of the quantum $\mathfrak{sp}_{2n}$ invariant. However, their formulation uses a *state sum* formula, which must be computed globally rather than locally. Local formulations allow for more efficient computation of link invariants, since local moves can greatly simplify a link diagram.

In [13], Kauffman introduces the *Kauffman polynomial*, a two-variable polynomial defined by

$$F(\mathcal{L})(a, z) = a^{-w(\mathcal{L}_\mathcal{D})} L(\mathcal{L}_\mathcal{D})$$

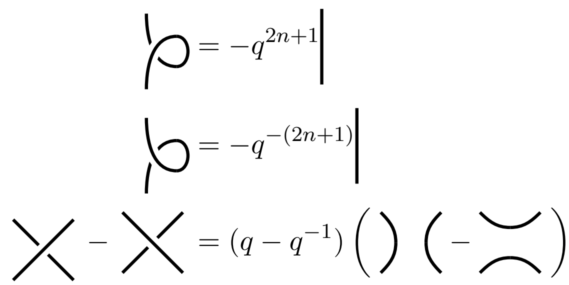
where $\mathcal{L}_\mathcal{D}$ is a diagram of a link $\mathcal{L}$, w is the writhe, and L is defined by being invariant under Reidemeister 2 and 3 moves, along with

$$L(O) = 1$$



where O denotes the unknot.

Since the braiding in $\mathbf{Web}(\mathfrak{sp}_{2n})$ is invariant under Reidemeister 2 and 3 moves, and



the braiding on the full subcategory of $\mathbf{Web}(\mathfrak{sp}_{2n})$ generated by 1 is a special case of the Kauffman polynomial (with some rescaling of generators).

CHAPTER 11

Further Work

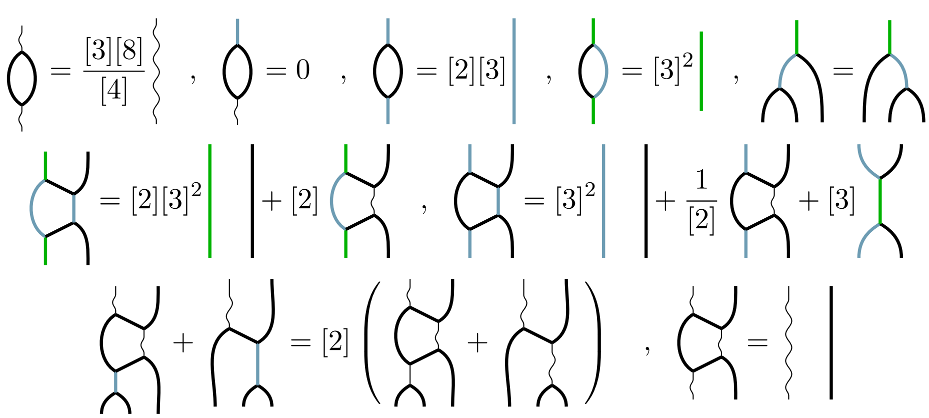
Of course, there is still much work to do. Some possible avenues of further work include:

- Prove Conjecture 3.0.6, that the functor $\Psi : \mathbf{Web}(\mathfrak{sp}_6) \to \mathbf{FundRep}(U_q(\mathfrak{sp}_6))$ is an equivalence of categories. We have proved that Ψ is full and essentially surjective, so we only lack showing that Ψ is faithful. One possible approach is to use the Bergman diamond lemma (for ring theory in [2] and upgraded to certain monoidal categories in [6]) to find a basis of endomorphism spaces of $\mathbf{Web}(\mathfrak{sp}_6)$, probably utilizing $\mathbf{Lad}(\mathfrak{sp}_6)$, and comparing dimensions with endomorphism spaces of $\mathbf{FundRep}(U_q(\mathfrak{sp}_6))$.

- Find enough relations in the general $\mathbf{Web}(\mathfrak{sp}_{2n})$ category to prove analogues of all our theorems for $\mathbf{Web}(\mathfrak{sp}_{2n})$. After these are found, we will have enough to be able to compute the type C_n link invariant with our local formulation for any n.

- After doing the above, prove that $\mathbf{Web}(\mathfrak{sp}_{2n})$ is equivalent to $\mathbf{FundRep}(U_q(\mathfrak{sp}_{2n}))$.

- Categorify everything in sight! A very rich area of research has been in categorifying link invariants. While the category $\mathbf{Web}(\mathfrak{sp}_6)$ does not seem amenable to categorification, where one would like all coefficients to be in $\mathbb{N}[q^{-1}, q]$, the category $\mathbf{Web}(\mathfrak{gsp}_6)$ seems like a start. The relations



(11.0.1)

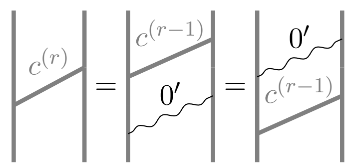
in Definition 6.0.2 have all positive coefficients, although there are denominators which seem troublesome. However, they are not so bad as they seem, remember that $\frac{[3][8]}{[4]} \in \mathbb{N}[q^{-1}, q]$, so the only "problematic denominators" are [2].

APPENDIX A

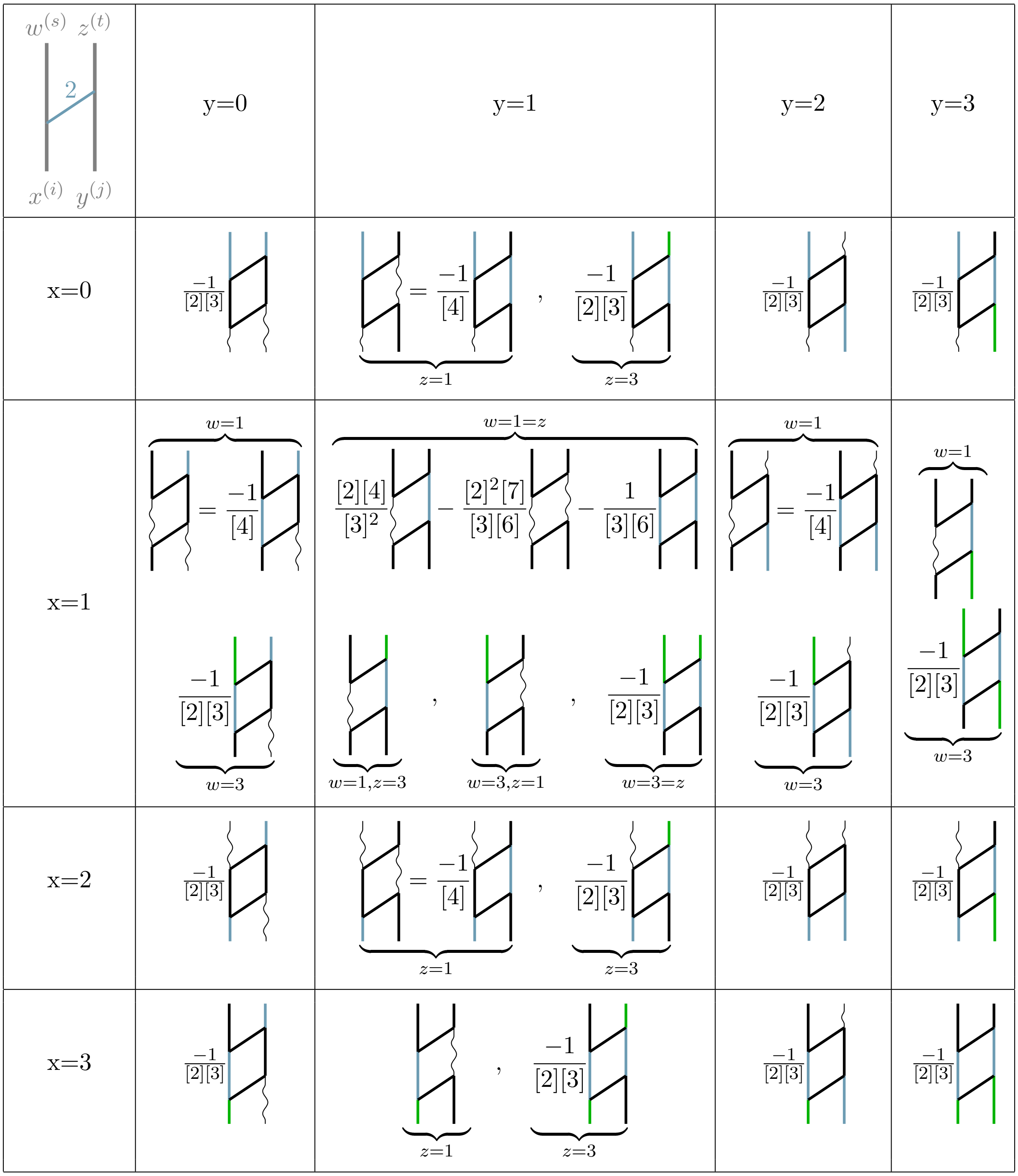
Appendix

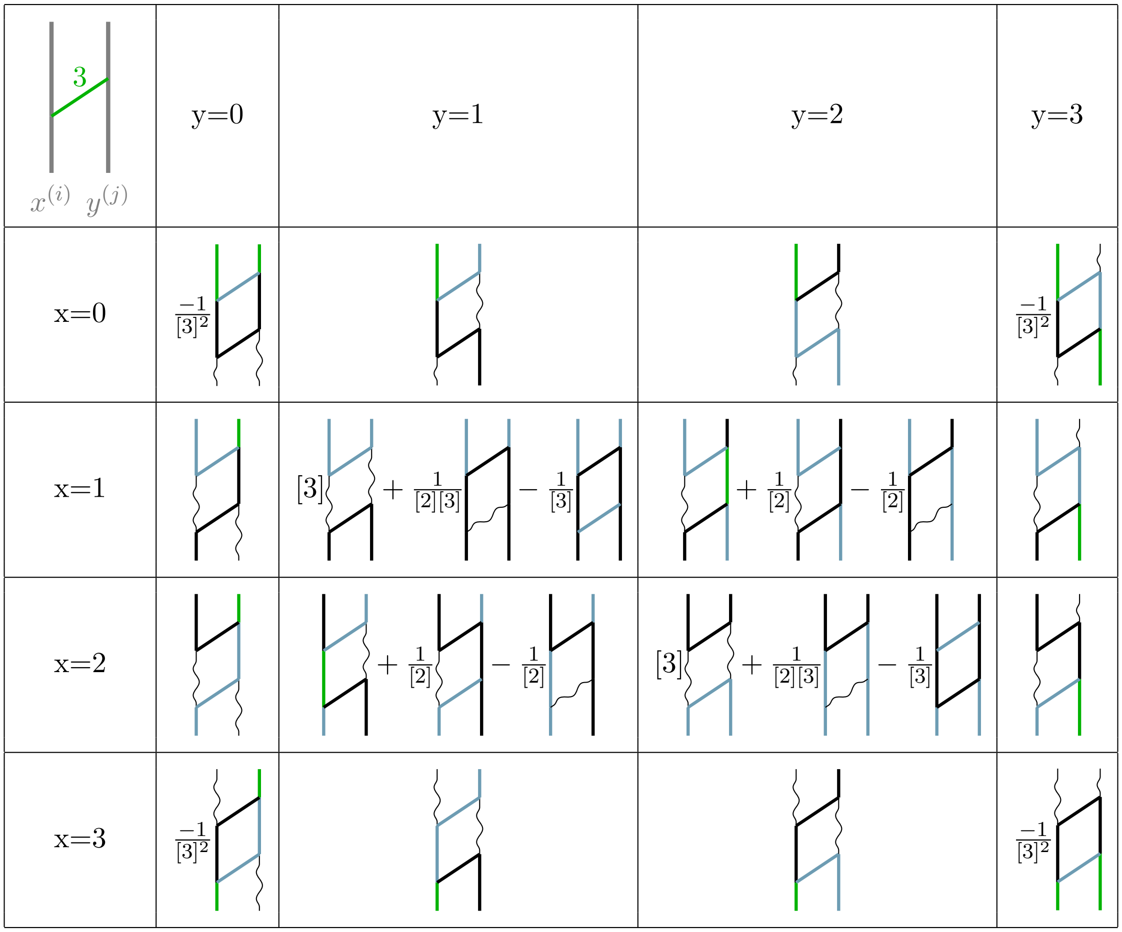
A.1 Relations in $\mathbf{Lad}(\mathfrak{sp}_6)$

The relations in $\mathbf{Lad}(\mathfrak{sp}_6)$ are as follows, together with those obtained from these via horizontal and vertical reflection. **Rung Explosion:** If $r \geq 1$, we have

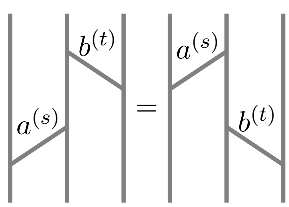


and for $r = 0$ and $c = 2$ or 3, the relations are recorded in the following tables:

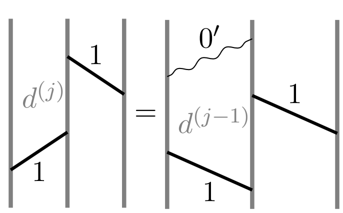




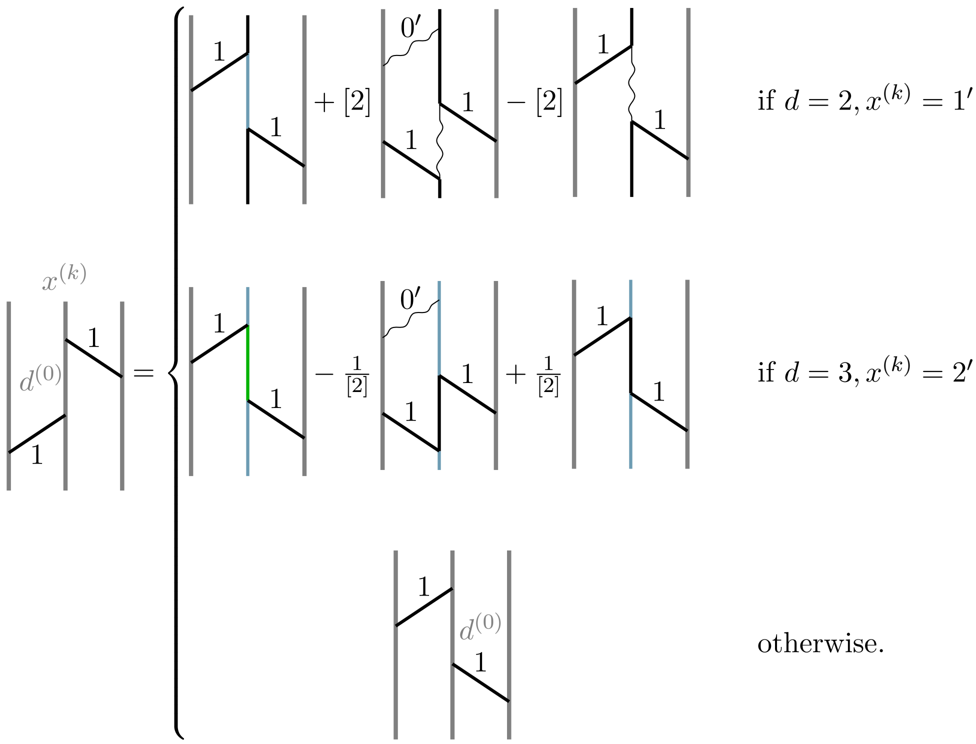
Rung Swap: If $a^{(s)} = 0'$ or $b^{(t)} = 0'$, then



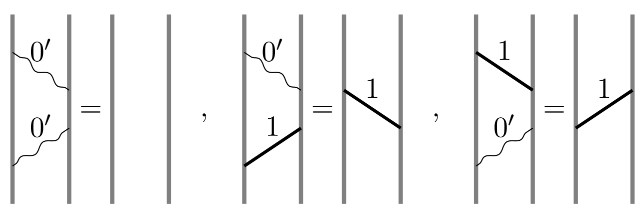
If $j > 0$, then



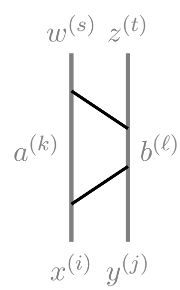
and otherwise



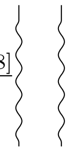
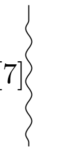
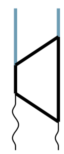
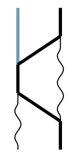
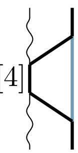
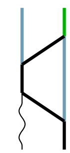
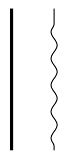
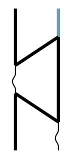
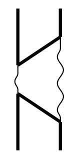
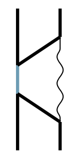
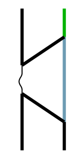
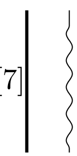
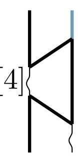
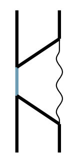
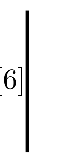
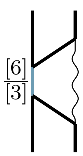
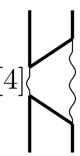
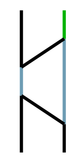
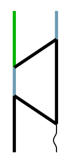
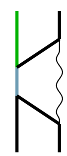
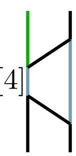
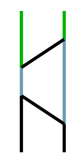
Square Relation: We have

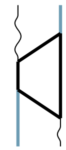
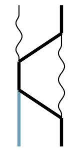
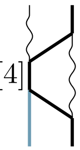
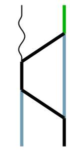
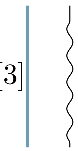
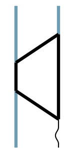
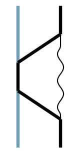
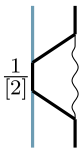
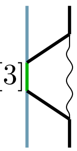
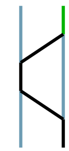
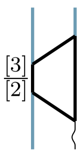
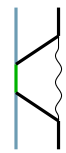
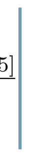
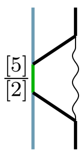
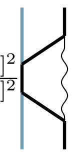
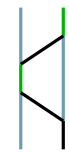
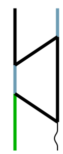
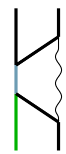
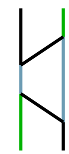
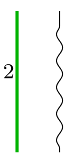
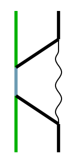
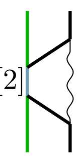


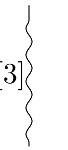
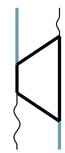
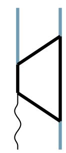
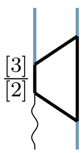
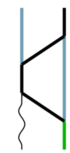
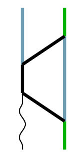
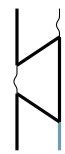
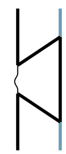
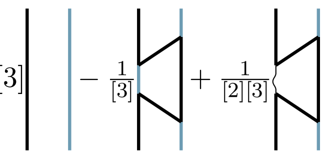
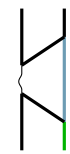
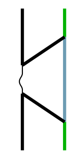
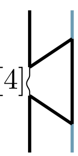
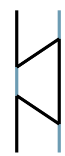
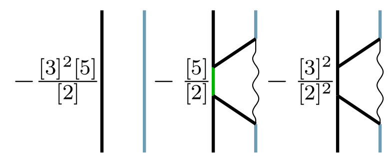
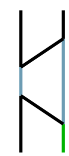
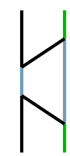
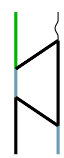
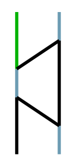
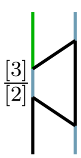
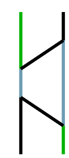
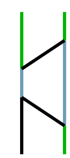
and the values of

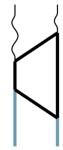
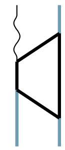
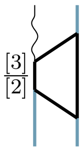
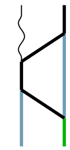
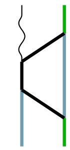
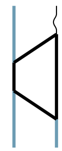
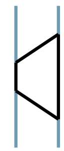
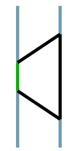
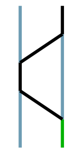
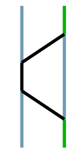
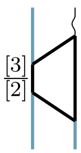
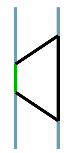
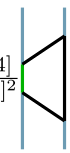
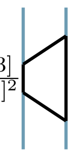
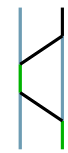
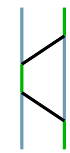
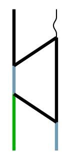
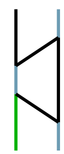
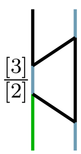
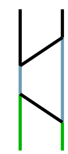
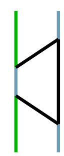
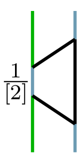
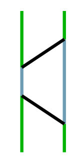


are recorded in the following tables:

	$(y,b,z)=(0,1,0)$	$(0,1,2)$	$(1,0,1)$	$(1,2,1)$	$(1,2,3)$
$(x,a,w)=(0,1,0)$	$\frac{-[3][8]}{[4]}$ 	0		$[2][7]$ 	0
$(0,1,2)$	0			$-[4]$ 	
$(1,0,1)$					
$(1,2,1)$	$[2][7]$ 	$-[4]$ 		$-[2][3][6]$  $-\frac{[6]}{[3]}$  $+[2][4]$ 	
$(1,2,3)$	0			$-[4]$ 	

	$(y,b,z)=(0,1,0)$	$(0,1,2)$	$(1,0,1)$	$(1,2,1)$	$(1,2,3)$
$(x,a,w)=(2,1,0)$	0			$-[4]$ 	
$(2,1,2)$	$-[2][3]$ 			$[3]^2$  $+\frac{1}{[2]}$  $-[3]$ 	
$(2,3,2)$	$\frac{[3][6]}{[2]}$ 	$\frac{[3]}{[2]}$ 		$-\frac{[3]^2[5]}{[2]}$  $-\frac{[5]}{[2]}$  $-\frac{[3]^2}{[2]^2}$ 	
$(3,2,1)$	0			$-[4]$ 	
$(3,2,3)$	$-[3]^2$ 	0		$[2][3]^2$  $+[2]$ 	0

	$(y,b,z)=(2,1,0)$	$(2,1,2)$	$(2,3,2)$	$(3,2,1)$	$(3,2,3)$
$(x,a,w)=(0,1,0)$	0	$-[2][3]$ 	$\frac{[3][6]}{[2]}$ 	0	$-[3]^2$ 
$(0,1,2)$			$\frac{[3]}{[2]}$ 		
$(1,0,1)$			$[3]$ 		
$(1,2,1)$	$-[4]$ 				
$(1,2,3)$			$\frac{[3]}{[2]}$ 		

	$(y,b,z)=(2,1,0)$	$(2,1,2)$	$(2,3,2)$	$(3,2,1)$	$(3,2,3)$
$(x,a,w)=(2,1,0)$			$\frac{[3]}{[2]}$ 		
$(2,1,2)$					
$(2,3,2)$	$\frac{[3]}{[2]}$ 		$-\frac{[3]^2[4]}{[2]}$  $+\frac{[4]}{[2]^2}$  $+\frac{[3]}{[2]^2}$ 		
$(3,2,1)$			$\frac{[3]}{[2]}$ 		0
$(3,2,3)$	0		$[3]^2$  $-\frac{1}{[2]}$ 	0	

REFERENCES

[1] A. Beliakova, Z. Guliyev, K. Habiro, and A. Lauda. Trace as an alternative decategorification functor. Acta Math. Vietnam., 39(4):425–480, 2014.

[2] G. Bergman. The diamond lemma for ring theory. Adv. Math., 1978.

[3] J. Birman and H. Wenzl. Braids, link polynomials and a new algebra. Trans. Amer. Math. Soc., 313(1):249–273, 1989.

[4] S. Cautis, J. Kamnitzer, and S. Morrison. Webs and quantum skew Howe duality. Math. Ann., 360(1-2):351–390, 2014. arXiv:1210.6437.

[5] V. Chari and A. Pressley. A Guide to Quantum Groups. Cambridge University Press, 1st edition, 1994.

[6] B. Elias. A diamond lemma for Hecke-type algebras. arXiv:1907:10571.

[7] B. Elias. Light ladders and clasp conjectures. arXiv:1510.06840.

[8] B. Elias. Quantum Satake in type A: part I. arXiv:1403.5570.

[9] W. Fulton and J. Harris. Representation Theory. Springer-Verlag New York, 1st edition, 2004.

[10] J. Hu. BMW algebra, quantized coordinate algebra and type C Schur-Weyl duality. Representation Theory, 15, 08 2007.

[11] J. Humphreys. Introduction to Lie Algebras and Representation Theory. Springer-Verlag New York, 1st edition, 1976.

[12] L.H. Kauffman. State models and the Jones polynomial. Topology, 26(3):395–407, 1987.

[13] L.H. Kauffman. An invariant of regular isotopy. Trans. Amer. Math. Soc., 318(2):417–471, 1990.

[14] M. Khovanov. A categorification of the Jones polynomial. Duke Math. J., 101(3):359–426, 2000. math.QA/9908171.

[15] D. Kim. Graphical calculus on representations of quantum Lie algebras. ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–University of California, Davis.

[16] G. Kuperberg. The quantum G_2 link invariants. Int. J. Math., 5(1):61–85, 1994.

[17] G. Kuperberg. Spiders for rank 2 Lie algebras. Comm. Math. Phys., 180(1):109–151, 1996. arXiv:9712003.

[18] W. B. R. Lickorish. An Introduction to Knot Theory. Springer-Verlag New York, 1st edition, 1997.

[19] S. Morrison. A diagrammatic category for the representation theory of ($U_q(sln)$). ProQuest LLC, Ann Arbor, MI, 2007. arXiv:0704.1503.

[20] H. Murakami and T. Ohtsuki. Quantum $sp(n)$ invariant of links via an invariant of colored planar graphs. Kobe J. Math., 1996.

[21] J. Murakami. The Kauffman polynomial of links and representation theory. Osaka J. Math., 24(4):745–758, 1987.

[22] K. Murasugi. The Jones polynomial and classical conjectures in knot theory. Topology, 26(2):187–194, 1987.

[23] H. Queffelec, D. E. V. Rose, and A. Sartori. Annular evaluation and link homology. arXiv:1802.04131.

[24] J. Rasmussen. Khovanov homology and the slice genus. Invent. Math., 182(2):419–447, 2010. arXiv:0402131.

[25] K. Reidemeister. Elementare begründung der knotentheorie. Abh. Amth Semin. Univ. Hambg., 1927.

[26] N. Yu. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. Comm. Math. Phys., 1:1–26, 1990.

[27] D. Rolfsen. Knots and Links. AMS Chelsea Publishing, 1st edition, 1976.

[28] D. E. V. Rose and L. Tatham. On webs in quantum type C. arXiv:2006.02491.

[29] Georg Rumer, Edward Teller, and Hermann Weyl. Eine fur die Valenztheorie geeignete Basis der binaren Vektorinvarianten. Nachr. Ges. Wiss. Gottingen Math. -Phys. Kl., pages 499–504, 1932.

[30] A. Sartori and D. Tubbenhauer. Webs and q-Howe dualities in types BCD. Trans. Amer. Math. Soc., 2018.

[31] P. Selinger. A survey of graphical languages for monoidal categories. In Bob Coecke, editor, New structures for Physics, volume 813 of Lecture Notes in Physics, pages 289–355. Springer, 2011.

[32] N. Snyder and P. Tingley. The half-twist for $u_q(\mathfrak{g})$ representations. Algebra & number theory, 3(7):809–834, 2009.

[33] M. B. Thistlethwaite. A spanning tree expansion of the Jones polynomial. Topology, 26(3):297–309, 1987.

[34] P. Tingley. A minus sign that used to annoy me but now i know why it is there. arXiv:1002.0555.

[35] A. Sossinsky V. Prasolov. Knots, Links, Braids, and 3-Manifolds: An Introduction to the New Invariants in Low-Dimensional Topology. American Mathematical Society, 1st edition, 1996.

[36] B. Xu. L-packets of quasisplit $gsp(2n)$ and $go(2n)$. Math.Ann., 2018.