Khovanov homology: what it is, why we like it, and how it's defined

Khovanov homology is an invariant of oriented links $\mathcal{L} \subseteq S^3$ taking values in bigraded $\mathbb{Z}$-modules that categorifies the **Jones polynomial**:

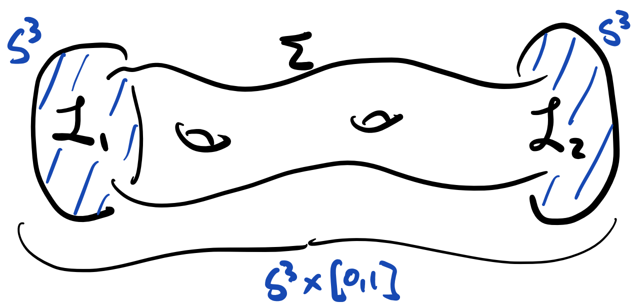
$$\begin{array}{ccc} & & Kh(\mathcal{L}) \in \mathbb{Z}\text{-mod}^{\mathbb{Z}\times\mathbb{Z}} \\ & \nearrow & \downarrow \dim_{q,t}(-\otimes\mathbb{Q})\big|_{t=-1} \\ S^3 \supseteq \mathcal{L} & \longmapsto & V_q(\mathcal{L}) \in \mathbb{Z}[q,q^{-1}] \end{array}$$

---

some easy reasons to like it:

① it is a strictly stronger invariant than Jones!
e.g. $V_q(5_1) = V_q(10_{132})$ but $Kh(5_1) \not\cong Kh(10_{132})$

② it is functorial under link cobordism:



$\Longrightarrow$ $Kh(\mathcal{L}_1) \xrightarrow{Kh(\Sigma)} Kh(\mathcal{L}_2)$

(gives applications to 4d topology)

③ connections to geometric/categorical representation theory and algebraic combinatorics.

<u>some algebraic conventions</u>:

$Kh(\mathcal{L})$ will be defined as the homology of a chain complex $C_{Kh}(\mathcal{D})$ of graded abelian groups associated to a diagram $\mathcal{D}$ of $\mathcal{L}$, i.e.

$$C_{Kh}(\mathcal{D}) = \Big(\bigoplus_t C^t(\mathcal{D}), \partial\Big) = \Big(\cdots \xrightarrow{\partial} C^t(\mathcal{D}) \xrightarrow{\partial} C^{t+1}(\mathcal{D}) \xrightarrow{\partial} \cdots\Big)$$

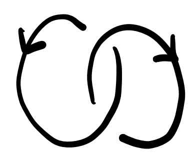
where $C^t(\mathcal{D}) = \bigoplus_q C^{q,t}(\mathcal{D})$.

thus $C_{Kh}(\mathcal{D}) = \Big(\bigoplus_{q,t} C^{q,t}(\mathcal{D}), \partial\Big)$, so $Kh(\mathcal{L}) = \bigoplus_{q,t} Kh^{q,t}(\mathcal{L})$

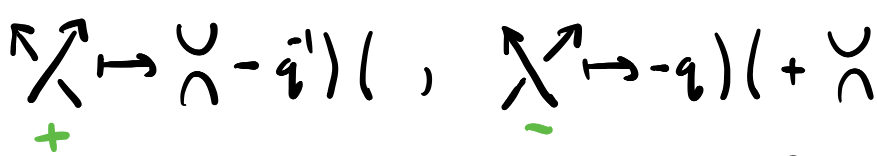
(∂ has "bidegree" $(0,1)$)

---

the definition is motivated by the Kauffman bracket description of the Jones polynomial:

- start with a diagram for $\mathcal{L}$, e.g. $\mathcal{D} =$ 

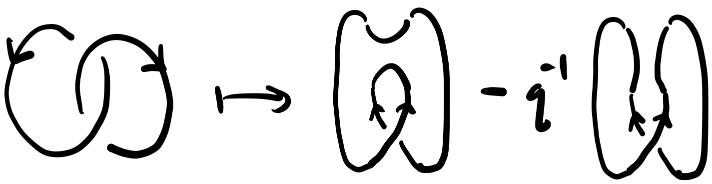
- apply the "Kauffman bracket" $\mathbb{Z}[q^{\pm}]$-linearly:

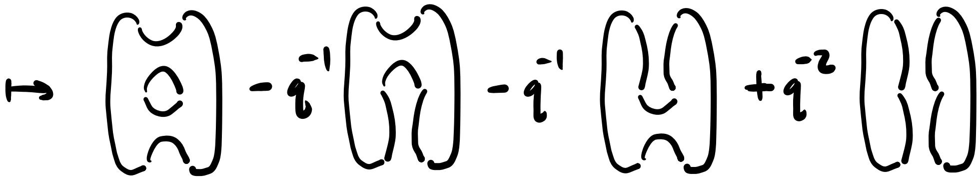


- evaluate circles via $\bigcirc = [2] := \dfrac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1}$

(• rescale by $(-q^2)^{w(\mathcal{D})}$ to account for framing)

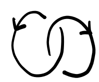
an example:





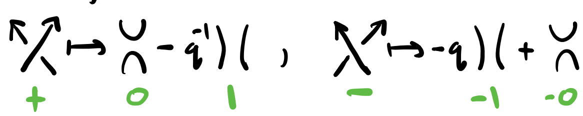
$$\longmapsto [2]^2 - 2q^{-1}[2] + q^{-2}[2]^2 = [2]\left(q + q^{-1} - 2q^{-1} + q^{-1} + q^{-3}\right)$$

$$= [2]\cdot(q+q^{-3}) \longmapsto (-q^2)^2[2]\cdot(q+q^{-3})$$

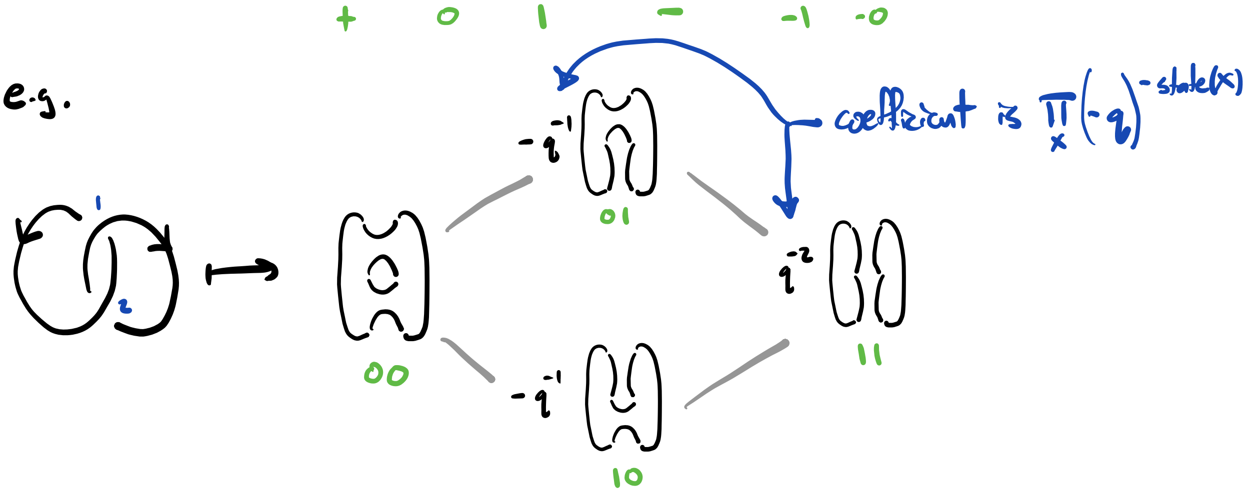
$\Rightarrow V_q($  $) = [2]\cdot(q^5+q)$

---

a repackaging: we can replace the first step with the "cube of resolutions", which is a weighted sum over "states" of the crossings:



e.g.

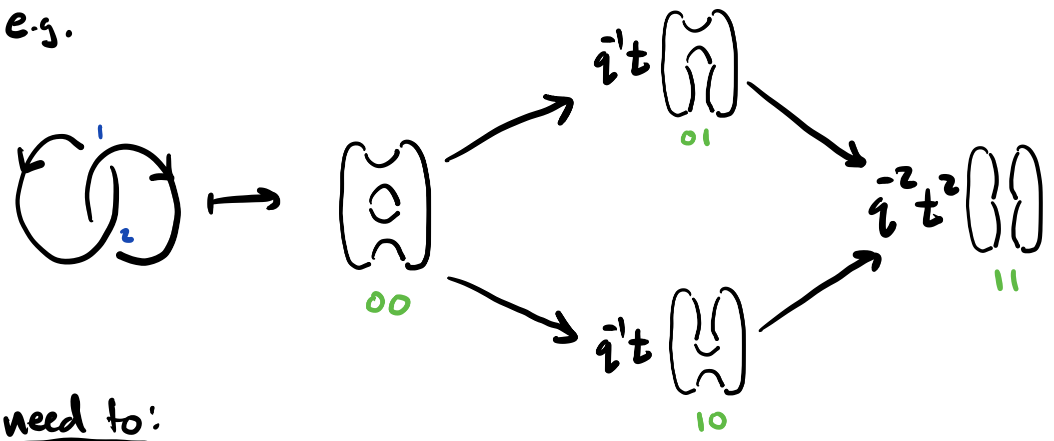


(then, assign [2] to circles, sum vertically, then horizontally)

Khovanov's deep insight: can reimagine this cube as a chain complex ($-q^{\pm 1} \rightsquigarrow q^{\pm} t^{\mp}$)

interpret as grading shift

e.g.



need to:

① assign ($\amalg$ circles) graded $\mathbb{Z}$-modules whose $\dim_q = \prod (q+q^{-1})$

($\Rightarrow \dim_{q,t}|_{t=-1}$ "=" Jones)

② assign $\mathbb{Z}$-module morphisms to arrows so that $\partial^2 = 0$

TQFT provides a solution to ① + ②:

- a (1+1)-dim TQFT (equivalently, a commutative Frobenius algebra) would give a commutative diagram, and a "sprinkling of signs" makes this a chain complex
- $H^*(\mathbb{CP}^1) \cong \mathbb{Z}[x]/x^2$ is a familiar commutative Frobenius algebra, and has $\dim_q(\mathbb{Z}[x]/x^2) = 1 + q^2$ if we set $\text{qdeg}(x) = 2$

grading shift down by 1

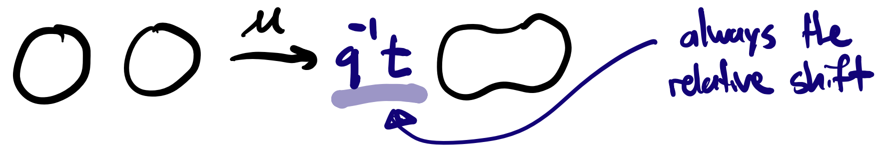
thus, let $\mathcal{U} = q^{-1}\,\mathbb{Z}[x]/x^2 \Rightarrow \dim_q(\mathcal{U}) = q^{-1} + q$

and send $\amalg$ circles $\longmapsto \bigotimes_{\text{circles}} \mathcal{U}$, e.g.

 $\mapsto \mathcal{U} \otimes_{\mathbb{Z}} \mathcal{U}$

along each edge of our cube, one of two things takes place:

- two circles can merge (and all others are left alone)



here, we assign the map $\mathcal{U}\otimes\mathcal{U} \longrightarrow q^{-1}t\,\mathcal{U}$ induced by multiplication:

total bidegree is (0,1)

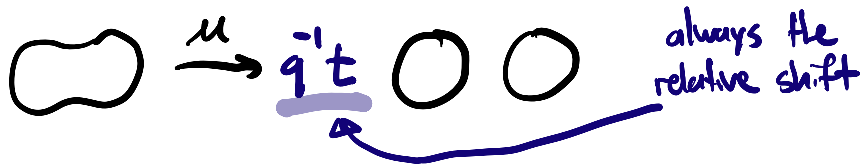
$$\mathcal{U}\otimes\mathcal{U} = q^{-2}\,\mathbb{Z}[x]/(x^2) \otimes \mathbb{Z}[x]/(x^2) \xrightarrow{\ \mu\ } q^{-2}t\,\mathbb{Z}[x]/(x^2) = q^{-1}t\,\mathcal{U}$$

$$1\otimes 1 \longmapsto 1$$
$$1\otimes x,\ x\otimes 1 \longmapsto x$$
$$x\otimes x \longmapsto 0$$

---

or,

- two circles can split (and all others are left alone)



here, we use the map $\mathcal{U} \longrightarrow q^{-1}t\,\mathcal{U}\otimes\mathcal{U}$ given by "comultiplication":

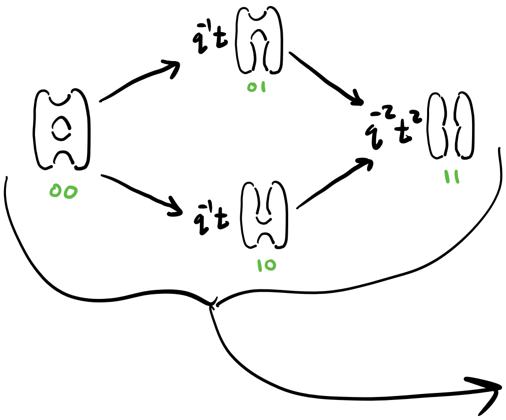
total bidegree is (0,1)

$$\mathcal{U} = q^{-1}\,\mathbb{Z}[x]/(x^2) \xrightarrow{\ \Delta\ } q^{-3}t\,\mathbb{Z}[x]/(x^2) \otimes \mathbb{Z}[x]/(x^2) = q^{-1}t\,\mathcal{U}\otimes\mathcal{U}$$

$$1 \longmapsto 1\otimes x + x\otimes 1$$

$$x \longmapsto x\otimes x$$

using these maps, our cube of resolutions becomes a commutative diagram (TQFT $\Rightarrow$ this), e.g.



$$\begin{array}{ccccc} & & q^{-1}t\,\mathcal{U} & & \\ & \overset{\mu}{\nearrow} & 01 & \overset{\Delta}{\searrow} & \\ \mathcal{U}\otimes\mathcal{U} & & \oplus & & q^{-2}t^{2}\,\mathcal{U}\otimes\mathcal{U} \\ 00 & \underset{\mu}{\searrow} & & \underset{-\Delta}{\nearrow} & 11 \\ & & q^{-1}t\,\mathcal{U} & & \\ & & 10 & & \end{array}$$

finally, we obtain a chain complex by multiplying each arrow by $(-1)^{\Sigma\left(\text{indices before the one that changes}\right)}$, and take $\oplus$ in each t-degree

---

$C_{Kh}(D)$ is then defined to be the resulting complex, after a shift of $(q^2t^{-1})^{w(D)}$

$$\begin{array}{ccccc} & & q^{-1}t\,\mathcal{U} & & \\ & \overset{\mu}{\nearrow} & 01 & \overset{\Delta}{\searrow} & \\ \mathcal{U}\otimes\mathcal{U} & & & & q^{-2}t^{2}\,\mathcal{U}\otimes\mathcal{U} \\ 00 & \underset{\mu}{\searrow} & & \underset{\Delta}{\nearrow} & 11 \\ & & q^{-1}t\,\mathcal{U} & & \\ & & 10 & & \end{array}$$

$$C_{Kh}(\text{Hopf link}) := q^4t^{-2}\,\mathcal{U}\otimes\mathcal{U} \xrightarrow{\begin{pmatrix}\mu\\ \mu\end{pmatrix}} t^{-1}q^{3}\,\mathcal{U}\oplus t^{-1}q^{3}\,\mathcal{U} \xrightarrow{(\Delta\ -\Delta)} q^{2}\,\mathcal{U}\otimes\mathcal{U}$$

since $\begin{pmatrix}\mu\\ \mu\end{pmatrix}$: $1\otimes1 \mapsto \begin{pmatrix}1\\1\end{pmatrix}$, $1\otimes x \mapsto \begin{pmatrix}x\\x\end{pmatrix}$, $x\otimes1 \mapsto \begin{pmatrix}x\\x\end{pmatrix}$, $x\otimes x \mapsto \begin{pmatrix}0\\0\end{pmatrix}$,

$(\Delta\ -\Delta)$: $\begin{pmatrix}1\\0\end{pmatrix} \mapsto 1\otimes x + x\otimes 1$

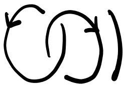
$\begin{pmatrix}0\\1\end{pmatrix} \mapsto -(1\otimes x + x\otimes 1)$

$\begin{pmatrix}x\\0\end{pmatrix} \mapsto x\otimes x$

$\begin{pmatrix}0\\x\end{pmatrix} \mapsto -x\otimes x$

we see that

$\mathrm{Kh}($$) = \left(q^4 t^{-2}\mathbb{Z} \oplus q^6 t^{-2}\mathbb{Z}\right) \oplus 0 \oplus \left(\mathbb{Z} \oplus q^2\mathbb{Z}\right)$

$\Rightarrow \dim_{q,t}\big(\mathrm{Kh}($$) \otimes \mathbb{Q}\big) = (q^4+q^6)t^{-2} + (1+q^2)$

@ $t=-1$, this gives $q^4+q^6+1+q^2 = [2](q^5+q) = V_q($$)$

of course, we must prove

<u>thm</u> [Khovanov, ...]: $\mathrm{Kh}(\mathcal{L})$ is an invariant of $\mathcal{L} \subseteq S^3$

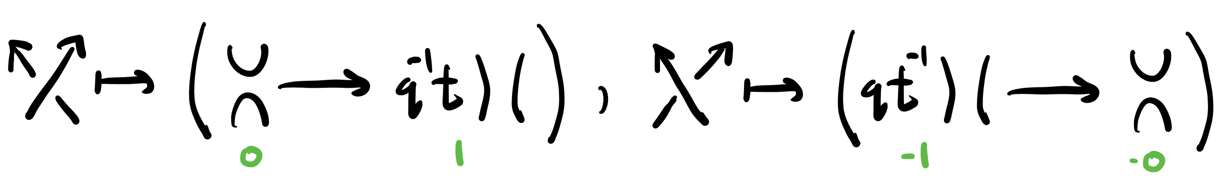
which involves showing that Reidemeister moves between link diagrams $\mathcal{D}_1 \sim \mathcal{D}_2$ induce (coherent) homotopy equivalences $C_{Kh}(\mathcal{D}_1) \simeq C_{Kh}(\mathcal{D}_2)$ (this is best done with the diagrammatically-local definition we'll see next...)

---

to summarize the construction of $\mathrm{Kh}(\mathcal{L})$

- choose a diagram $\mathcal{D}$ for the link $\mathcal{L}$

- form the cube of resolutions using the rules:



- assign $\bigotimes_{\text{circles}} \mathcal{U}$ to each vertex of the cube and μ or Δ $(\otimes \mathrm{Id} \otimes \mathrm{Id} \otimes \cdots)$ to each edge, as appropriate

---

- sprinkle signs, $\oplus$ in t-degree, shift by $(q^2t^{-1})^{w(\mathcal{D})}$ to obtain $C_{Kh}(\mathcal{D})$
- and take homology. congrats, you can now compute Khovanov homology :)

next up: more examples + properties ...